Finite embeddability via nuclear and conuclear images

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GALATOS & JIPSEN 2013

For each variety of **integral** residuated lattices:

axiomatized by equations in the signature $\{\vee, \cdot, 1\} \implies$ FEP.

GALATOS & JIPSEN 2017

For each variety of **distributive integral** residuated lattices:

axiomatized by equations in the signature $\{\wedge, \vee, \cdot, 1\} \implies$ FEP.

A class of algebras K has the **Finite Embeddability Property** (**FEP**) if each universal sentence which fails in some K-algebra fails in a finite K-algebra.

A **finitely axiomatizable** universal class with the FEP has a decidable universal theory (either a finite proof or a finite counter-example).

We prove these results:

- at the level of ordered universal algebra,
- where both results are instances of a single theorem,
- using nuclei and conuclei rather than residuated frames,
- thus explaining the semantic significance of the syntactic restriction,
- more generally for positive universal classes (e.g. linear),
- and therefore for more varieties (e.g. semi-linear).

Limitation: not clear how to go beyond the integral case.

In **ordered universal algebra**, there are interesting unexplored fragments, namely: restrict atomic formulas to $t(x_1, \ldots, x_n) \leq y$.

Partially ordered algebras

A **po-algebra** is an algebra $\mathbf{A}$ (in some **base signature**) with a partial order $\leq$ such that for each n-ary primitive operation $\circ^{\mathbf{A}}$

$$a_1 \leq b_1 \ \& \ \ldots \ \& \ a_n \leq b_n \implies \circ^{\mathbf{A}}(a_1, \ldots, a_n) \leq \circ^{\mathbf{A}}(b_1, \ldots, b_n).$$

A po-algebra $\mathbf{A}$ is **integral** if for each n-ary primitive operation $\circ^{\mathbf{A}}$

$$\circ^{\mathbf{A}}(a_1, \ldots, a_n) \leq a_i \qquad \text{for each } i \in \{1, \ldots, n\}.$$

An **sℓ-algebra** is an algebra $\mathbf{A}$ with a join semilattice operation $\vee$ such that

$$\circ^{\mathbf{A}}(a_1, \ldots, b \vee c, \ldots, a_n) = \circ^{\mathbf{A}}(a_1, \ldots, b, \ldots, a_n) \vee \circ^{\mathbf{A}}(a_1, \ldots, c, \ldots, a_n).$$

Its **po-algebra reduct** is the algebra $\mathbf{A}$ with the partial order induced by $\vee$.

An sℓ-algebra is **integral** if its po-algebra reduct is an integral po-algebra.

An **ℓ-algebra** is an sℓ-algebra with a $\wedge$ operation which makes it a lattice.

Example. A **po-monoid** is a monoid **A** with a partial order $\leq$ such that

$$a_1 \leq b_1 \;\&\; a_2 \leq b_2 \implies a_1 \cdot a_2 \leq b_1 \cdot b_2.$$

An **sℓ-monoid** is a monoid **A** with a join semilattice operation $\vee$ such that

$$(a_1 \vee a_2) \cdot b = (a_1 \cdot b) \vee (a_2 \cdot b), \qquad a \cdot (b_1 \vee b_2) = (a \cdot b_1) \vee (a \cdot b_2).$$

An **ℓ-monoid** is an sℓ-monoid with a $\wedge$ operation making it a lattice.

A po-monoid is integral if and only if 1 is its top element.

Example. Instead of monoids, consider **$\wedge$-monoids**: algebras $\langle A, \wedge, \cdot, 1\rangle$ such that $\langle A, \wedge\rangle$ is a meet semilattice and $\langle A, \cdot, 1\rangle$ is a monoid.

In each **integral po-($\wedge$-monoid)** the operation $\wedge$ is a meet w.r.t. $\leq$:

$$a \wedge b \leq a, \quad a \wedge b \leq b, \qquad x \leq a \text{ and } x \leq b \implies x = x \wedge x \leq a \wedge b.$$

An **integral sℓ-($\wedge$-monoid)** is the same thing as an **integral ℓ-($\wedge$-monoid)**, which is the same thing as a **distributive integral ℓ-monoid**, in the sense:

$\langle A, \wedge, \vee, \cdot, 1\rangle$ with $\langle A, \wedge, \vee\rangle$ distributive and $\langle A, \vee, \cdot, 1\rangle$ an integral sℓ-monoid.

That is, instead of treating distributive ℓ-monoids as

integral ℓ-algebras satisfying an extra equation,

we treat them as

integral (s)ℓ-algebras, just in a different base signature.

The **residual** (in a given position) of an n-ary operation $\circ^{\mathbf{A}}$ of an po-algebra $\mathbf{A}$ is, if it exists, the n-ary operation $\backslash\!\!\backslash^{\mathbf{A}}$ determined by the equivalence

$$\circ^{\mathbf{A}}(a_1, \ldots, b, \ldots, a_n) \leq c \iff b \leq \backslash\!\!\backslash^{\mathbf{A}}(a_1, \ldots, c, \ldots, a_n).$$

A **partially residuated po-algebra** (of a given **residuation type**) is an expansion of a po-algebra by some of the residuals of its operations.

A **residuated po-algebra** is an expansion of a po-algebra by all residuals.

EXAMPLES: residuated pomonoids, left-residuated pomonoids, pomonoids:

$$a \leq c/b \iff a \cdot b \leq c \iff b \leq a\backslash c.$$

FACT. The operation $\circ^{\mathbf{A}}$ has a residual (in a given position) if and only if the following joins exist and are preserved by $\circ^{\mathbf{A}}$ in the given position:

$$\bigvee\{x \in \mathbf{A} \mid \circ^{\mathbf{A}}(a_1, \ldots, x, \ldots, a_n) \leq c\}.$$

FACT. TFAE for each join semilattice:

- There are no infinite ascending chains.
- Each non-empty ideal is a principal ideal.
- Each non-empty join exists and coincides with a finite subjoin.

Such join semilattices will be called **join-finite**.

FACT. Each integral **join-finite** sℓ-algebra is a residuated ℓ-algebra.

Nuclei and conuclei

A **conuclease** on a po-algebra **A** is an interior operator σ on **A**, i.e.

$$\sigma\sigma a = \sigma a \leq a, \qquad\qquad a \leq b \implies \sigma a \leq \sigma b,$$

such that for each n-ary primitive (or equivalently, term) operation $\circ^{\mathbf{A}}$ of **A**

$$\circ^{\mathbf{A}}(\sigma a_1, \ldots, \sigma a_n) \leq \sigma(\circ^{\mathbf{A}}(a_1, \ldots, a_n)).$$

EXAMPLE. In the case of po-monoids: $\sigma a \cdot \sigma b \leq \sigma(a \cdot b), \quad 1 \leq \sigma 1.$

In the case of po-($\wedge$-monoids) also: $\sigma a \wedge \sigma b \leq \sigma(a \wedge b).$

FACT. An interior operator σ on a po-algebra **A** is a conucleus if and only if its image is a subalgebra of **A**:

$$\mathbf{A}_\sigma := \sigma[\mathbf{A}] = \{a \in \mathbf{A} \mid \sigma a = a\}.$$

The set of all **conuclear images** $\mathbf{A}_\sigma$ of **A** will be denoted by $\mathbb{C}(\mathbf{A})$.

A **partial conucleus** on a po-algebra **A** is a **partial interior operator** on **A**, i.e. an interior operator σ **on an upset** of **A**, such that

$$\circ^{\mathbf{A}}(\sigma a_1, \dots, \sigma a_n) \leq \sigma(\circ^{\mathbf{A}}(a_1, \dots, a_n))$$

in the sense that if the left-hand side exists, so does the right-hand side.

Equivalently, this is a conucleus on an upward closed subalgebra of **A**.

FACT. A partial interior operator σ on a po-algebra **A** is a partial conucleus if and only if its image is a subalgebra of **A**:

$$\mathbf{A}_\sigma := \sigma[\mathbf{A}] = \{a \in \mathbf{A} \mid \sigma a = a\}.$$

FACT. If $\mathbf{A}$ is an sℓ-algebra (an ℓ-algebra), then so is $\mathbf{A}_\sigma$:

$$a \vee^{\mathbf{A}_\sigma} b := a \vee b, \qquad\qquad a \wedge^{\mathbf{A}_\sigma} b := \sigma(a \wedge b).$$

If $\mathbf{A}$ is partially residuated, then so is $\mathbf{A}_\sigma$:

$$\backslash\!\!\backslash^{\mathbf{A}_\sigma}(a_1, \ldots, a_n) := \sigma(\backslash\!\!\backslash^{\mathbf{A}}(a_1, \ldots, a_n)).$$

FACT. For each partial conucleus σ on a partially residuated (s)ℓ-algebra $\mathbf{A}$

$$a_1 \wedge^{\mathbf{A}} a_2 = b \text{ for } a_1, a_2, b \in \mathbf{A}_\sigma \implies a_1 \wedge^{\mathbf{A}_\sigma} a_2 = b,$$
$$\backslash\!\!\backslash^{\mathbf{A}}(a_1, \ldots, a_n) = b \text{ for } a_1, \ldots, a_n, b \in \mathbf{A}_\sigma \implies \backslash\!\!\backslash^{\mathbf{A}_\sigma}(a_1, \ldots, a_n) = b.$$

That is, $\mathbf{A}_\sigma$ is a partial subalgebra of $\mathbf{A}$.

A **nucleus** on a po-algebra **A** is a closure operator γ on **A**, i.e.

$$a \leq \gamma a = \gamma\gamma a, \qquad\qquad a \leq b \implies \gamma a \leq \gamma b,$$

such that for each n-ary primitive (or equivalently, term) operation $\circ^{\mathbf{A}}$ of **A**

$$\circ^{\mathbf{A}}(\gamma a_1, \ldots, \gamma a_n) \leq \gamma(\circ^{\mathbf{A}}(a_1, \ldots, a_n)).$$

EXAMPLE. In the case of po-monoids: $\gamma a \cdot \gamma b \leq \gamma(a \cdot b)$.

In the case of po-($\wedge$-monoids) also: $\gamma a \wedge \gamma b \leq \gamma(a \wedge b)$.

The **nuclear image** $\mathbf{A}_\gamma$ of **A** is the po-algebra over the universe

$$\gamma[\mathbf{A}] = \{a \in \mathbf{A} \mid \gamma a = a\}$$

with the operations $\circ^{\mathbf{A}_\gamma}(a_1, \ldots, a_n) := \gamma(\circ^{\mathbf{A}}(a_1, \ldots, a_n))$.

The set of nuclear images of **A** will be denoted by $\mathbb{N}(\mathbf{A})$.

Fact. If $\mathbf{A}$ is an sℓ-algebra (an ℓ-algebra), then so is $\mathbf{A}_\gamma$:

$$a \vee^{\mathbf{A}_\gamma} b := \gamma(a \vee b), \qquad a \wedge^{\mathbf{A}_\gamma} b := a \wedge b.$$

If $\mathbf{A}$ is partially residuated, then so is $\mathbf{A}_\gamma$:

$$\backslash\!\!\backslash^{\mathbf{A}_\gamma}(a_1, \ldots, a_n) := \backslash\!\!\backslash^{\mathbf{A}}(a_1, \ldots, a_n).$$

Fact. Let γ be a nucleus on an sℓ-algebra $\mathbf{A}$. Then

$$a_1 \vee^{\mathbf{A}} a_2 = b \text{ for } a_1, a_2, b \in \mathbf{A}_\gamma \implies a_1 \vee^{\mathbf{A}_\gamma} a_2 = b,$$

$$\circ^{\mathbf{A}}(a_1, \ldots, a_n) = b \text{ for } a_1, \ldots, a_n, b \in \mathbf{A}_\gamma \implies \circ^{\mathbf{A}_\gamma}(a_1, \ldots, a_n) = b.$$

That is, $\mathbf{A}_\gamma$ is a partial subalgebra of $\mathbf{A}_\gamma$.

Universal classes and the FEP

TFAE for each class of algebras (in a given signature):

- K is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}_{\mathrm{U}}$.
- K is axiomatized by universal sentences:

$$\forall \overline{x}\, \phi \qquad \text{where } \phi \text{ is quantifier-free.}$$

- K is axiomatized by sentences of the form

$$t_1 = u_1 \;\&\; \ldots \;\&\; t_m = u_1 \implies v_1 = w_1 \text{ or } \ldots \text{ or } v_n = w_n.$$

Example. $x \vee y = 1 \implies x = 1$ or $y = 1$ (1 is join irreducible).

The **universal class** generated by a class of algebras K is

$$\mathbb{U}(\mathsf{K}) := \mathbb{ISP}_{\mathrm{U}}(\mathsf{K}).$$

TFAE for each class of algebras (in a given signature):

- K is closed under $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}_\mathrm{U}$.
- K is axiomatized by positive universal sentences:

  $\forall \bar{x}\, \phi$ where ϕ is quantifier-free and **positive** (only $\wedge$ and $\vee$).
- K is axiomatized by sentences of the form

$$t_1 = u_1 \text{ or } \dots \text{ or } t_n = u_n.$$

EXAMPLE. $x \leq y$ or $y \leq x$ (linearity).

The **positive universal class** generated by a class of algebras K is

$$\mathbb{HSP}_\mathrm{U}(\mathsf{K}).$$

Theorem (Maltsev 1973)

Consider an algebra $\mathbf{A}$ and a class of algebras K. If for each finite set $X \subseteq \mathbf{A}$ the partial subalgebra $\mathbf{A}|_X$ embeds into some algebra in K, then $\mathbf{A} \in \mathbb{U}(\mathsf{K})$.

Given an algebra $\mathbf{A}$ and a set $X \subseteq \mathbf{A}$, an injective map $h\colon X \to \mathbf{B}$ is an **embedding** of the **partial subalgebra $\mathbf{A}|_X$** into an algebra $\mathbf{B}$ if:

$$\circ^{\mathbf{A}}(a_1, \ldots, a_n) = b \text{ for } a_1, \ldots, a_n, b \in X \implies \circ^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = h(b).$$

A universal class K has the **Finite Embeddability Property** (**FEP**) if each finite partial subalgebra of each $\mathbf{A} \in \mathsf{K}$ embeds into a finite algebra in K.

FEP for integral algebras

Finite embeddability for integral algebras

Each universal class of **integral** partially residuated (s)ℓ-algebras which is closed under **nuclear** and **conuclear** images has the FEP.

Corollary for distributive integral algebras

Each universal class of **distributive integral** partially residuated ℓ-algebras which is closed under $\wedge$**-nuclear** and $\wedge$**-conuclear** images has the FEP.

Special case. This subsumes the two results of Galatos & Jipsen.

Finite embeddability for integral algebras

Each universal class of integral partially residuated (s)ℓ-algebras which is closed under nuclear and conuclear images has the FEP.

More precisely...

Let $\mathbf{A}$ be an integral partially residuated (s)ℓ-algebra. Then

$$\mathbf{A} \in \mathbb{U}(\textbf{finite} \text{ nuclear images of partial conuclear images of } \mathbf{A}).$$

Proof

For each finite $X \subseteq \mathbf{A}$ we embed $\mathbf{A}|_X$ into $\mathbb{N}_{\text{fin}}\mathbb{C}_{\text{part}}(\mathbf{A})$.

Let σ be the partial conucleus on $\mathbf{A}$ generated by $X \subseteq \mathbf{A}$.

Let γ be the nucleus on $\mathbf{A}_\sigma$ generated by $X \subseteq \mathbf{A}_\sigma$.

We have seen that $\mathbf{A}|_X \hookrightarrow \mathbf{A}_\sigma$ and $\mathbf{A}_\sigma|_X \hookrightarrow (\mathbf{A}_\sigma)_\gamma$ (as partial subalgebras).

We only need to show that $(\mathbf{A}_\sigma)_\gamma$ is finite.

HIGMAN (1952)

TFAE for each poset:

- In each sequence of elements $(x_i)_{i\in\omega}$ there are $i < j$ with $x_i \geq x_j$.
- There are no infinite antichains and no infinite ascending chains.
- Each infinite sequence has an infinite descending or constant subseq.
- There is no infinite ascending chain of downsets.
- Each downset is the downward closure of a finite set.

Such a poset will be called **dually well partially ordered** (**dwpo**).

TRICHOTOMY FOR INFINITE POSETS. Each infinite poset has an infinite antichain or an infinite ascending chain or an infinite descending chain.

Higman's Lemma (1952)

Let **A** be an integral po-algebra in a finite signature generated by $X \subseteq \mathbf{A}$.
If X is dually well partially ordered (e.g. if X is finite), then **A** is **dwpo**.

Corollary for sℓ-algebras (Galatos & Horčík 2013)

Let **A** be an integral sℓ-algebra in a finite signature generated by $X \subseteq \mathbf{A}$.
If X is dually well partially ordered (e.g. if X is finite), then **A** is **join-finite**.

Partial conucleus generated by a finite set

Consider an integral sℓ-algebra **A** in a finite signature and a finite set $X \subseteq \mathbf{A}$.
The partial conucleus σ_X on **A generated by** X has $\mathrm{Sg}^{\mathbf{A}} X$ as its image.

Description of nuclear images

A closure operator γ on a **residuated** po-algebra $\mathbf{A}$ is a nucleus if and only if its image $\mathbf{A}_\gamma$ is a **residuation ideal** of $\mathbf{A}$:

$$x \in \mathbf{A}_\gamma \text{ and } a_1, \ldots, a_n \in \mathbf{A} \implies \mathbin{\backslash\!\backslash}^{\mathbf{A}}(a_1, \ldots, x, \ldots, a_n) \in \mathbf{A}_\gamma.$$

Example. For residuated pomonoids: $x \in \mathbf{A}_\gamma,\, a \in \mathbf{A} \implies a\backslash x, x/a \in \mathbf{A}_\gamma$.

The closure operator on a join-finite semilattice induced by X is defined as:

$$\gamma a := \bigwedge \{x \in X \mid a \leq x\}.$$

Nucleus generated by a set

Consider an integral join-finite sℓ-algebra $\mathbf{A}$ in a finite signature and $X \subseteq \mathbf{A}$. The nucleus **generated by** X is the closure operator σ_X on $\mathbf{A}$ induced by the residuation ideal of $\mathbf{A}$ generated by X.

THE ONLY NEW PART OF THE PROOF

Let **A** be an integral $s\ell$-algebra generated by a finite set $X \subseteq \mathbf{A}$.
Then the nucleus generated by X has a finite image.

PROOF. Let **B** be the (dwpo) sub-po-algebra of **A** generated by X.

Each element of **A** is a non-empty finite join of elements of **B**.

The residuation ideal generated by X induces the same closure operator as the residuation ideal **with respect to B** generated by X:

$$x \in \mathbf{A}_\gamma \text{ and } b_1, \ldots, b_n \in \mathbf{B} \implies \backslash\!\!\backslash^{\mathbf{A}}(b_1, \ldots, x, \ldots, b_n) = \backslash\!\!\backslash^{\mathbf{A}}(\overline{b}, x) \in \mathbf{A}_\gamma.$$

Why? Each $a \in \mathbf{A}$ is a finite join of elements $b \in \mathbf{A}$, therefore each $\backslash\!\!\backslash^{\mathbf{A}}(\overline{a}, x)$ is a finite meet of elements of the form $\backslash\!\!\backslash^{\mathbf{A}}(\overline{b}, x)$.

Take $X_0 := X$ and define X_{n+1} as the set consisting of elements of the form

$$\backslash\!\backslash^{\mathsf{A}}(\overline{b}, x) \qquad \text{for all } x \in X_n \text{ and } \overline{b} \in \mathbf{B}^{n-1}.$$

Each of these elements lies above $x \in X_n$ by integrality.

Think of this as growing a tree above each element $x \in X$.

If each stage X_n is finite, **then** the process stops in finitely many steps: otherwise, **Kőnig's lemma** yields an infinite increasing chain in **A**.

Kőnig's lemma

Each finitely branching tree with an infinite height has an infinite branch.

Key lemma

The following set is finite for each $x \in \mathbf{A}$ and each residual $\backslash\!\!\backslash^{\mathbf{A}}$:

$$\{\backslash\!\!\backslash^{\mathbf{A}}(\overline{a}, x) \mid \overline{a} \in \mathbf{B}^{n-1}\}.$$

Proof. $\mathbf{B}^{n-1}$ is dwpo because a finite product of dwpo sets is dwpo.

The above set is a **wpo** (dual of dwpo): given a sequence

$$\backslash\!\!\backslash^{\mathbf{A}}(\overline{a}_1, x),\ \backslash\!\!\backslash^{\mathbf{A}}(\overline{a}_2, x),\ \ldots,$$

there are $i < j$ with $\overline{a}_i \geq \overline{a}_j$, so there are $i < j$ with $\backslash\!\!\backslash^{\mathbf{A}}(\overline{a}_i, x) \leq \backslash\!\!\backslash^{\mathbf{A}}(\overline{a}_j, x)$. In other words, the set has no infinite antichains or descending chains.

Moreover, the set lives in $\mathbf{A}$, which has no infinite ascending chains.

So, by the trichotomy for infinite posets, the set must be finite.

Finite embeddability for integral algebras

Each **universal class** of integral partially residuated (s)ℓ-algebras which is closed under **nuclear** and **conuclear** images has the FEP.

What are these classes in syntactic terms?

Nuclear universal classes: integral case

A **nuclear class** of $s\ell$-algebras is a class closed under nuclear images.

THEOREM

TFAE for each universal class K of **integral** $s\ell$-algebras:

- K is closed under $\mathbb{N}$, i.e. it is a nuclear universal class.
- K is closed under $\mathbb{H}$, i.e. it is a positive universal class.
- K is axiomatized by a set of positive universal sentences.

COROLLARY

TFAE for each quasivariety K of **integral** $s\ell$-algebras:

- K is closed under $\mathbb{N}$, i.e. it is a nuclear quasivariety.
- K is closed under $\mathbb{H}$, i.e. it is a variety.
- K is axiomatized by a set of equations.

PROOF.

FACT

Each nuclear image $\mathbf{A}_\gamma$ of an $s\ell$-algebra $\mathbf{A}$ is a homomorphic image of $\mathbf{A}$:

$$\gamma(t^{\mathbf{A}}(a_1,\ldots,a_n)) = \gamma(t^{\mathbf{A}}(\gamma a_1,\ldots,\gamma a_n)) = t^{\mathbf{A}_\gamma}(\gamma a_1,\ldots,\gamma a_n),$$

That is, $\mathbb{N}(\mathbf{A}) \subseteq \mathbb{H}(\mathbf{A})$.

PROOF. $\gamma(t^{\mathbf{A}}(a_1,\ldots,a_n)) \leq \gamma(t^{\mathbf{A}}(\gamma a_1,\ldots,\gamma a_n))$ because $a_i \leq \gamma a_i$, and

$$\gamma(t^{\mathbf{A}}(\gamma a_1,\ldots,\gamma a_n)) \leq \gamma(\gamma t^{\mathbf{A}}(a_1,\ldots,a_n)) = \gamma(t^{\mathbf{A}}(a_1,\ldots,a_n)).$$

FACT

Let $\mathbf{A}$ be a **finitely generated integral** $s\ell$-algebra. Then each quotient $\mathbf{A}/\theta$ is isomorphic to a nuclear image $\mathbf{A}_\gamma$ for

$$\gamma a := \max[a]_\theta.$$

That is, $\mathbb{H}(\mathbf{A}) \subseteq \mathbb{IN}(\mathbf{A})$.

PROOF. $\max[a]_\theta = \bigvee[a]_\theta$ exists because $\mathbf{A}$ is join-finite. Moreover,

$$[a_1]_\theta = [\gamma a_1]_\theta, \ldots, [a_n]_\theta = [\gamma a_n]_\theta \text{ for all } a_1, \ldots, a_n \in \mathbf{A},$$

so

$$[\circ^{\mathbf{A}}(\gamma a_1, \ldots, \gamma a_n)]_\theta = [\circ^{\mathbf{A}}(a_1, \ldots, a_n)]_\theta,$$

and therefore

$$\circ^{\mathbf{A}}(\gamma a_1, \ldots, \gamma a_n) \leq^{\mathbf{A}} \max[\circ^{\mathbf{A}}(a_1, \ldots, a_n)]_\theta = \gamma(\circ^{\mathbf{A}}(a_1, \ldots, a_n)).$$

It only remains to observe that $\mathbb{H}(\mathbf{A}) \subseteq \mathbb{U}(\mathbb{HS}_{\mathrm{fg}}(\mathbf{A})) \subseteq \mathbb{U}(\mathbb{NS}(\mathbf{A}))$. □

FACT

The nuclear universal class generated by a class of sℓ-algebras K is

$$\mathbb{INSP}_{\mathrm{U}}(\mathsf{K}).$$

In the **integral** case, this class coincides with $\mathbb{HSP}_{\mathrm{U}}(\mathsf{K})$.

FACT

The nuclear quasivariety generated by a class of sℓ-algebras K is

$$\mathbb{INSPP}_{\mathrm{U}}(\mathsf{K}).$$

In the **integral** case, this class coincides with $\mathbb{HSP}(\mathsf{K})$.

Nuclear universal classes: beyond the integral case

Nuclear quasivarieties need not be varieties in general.

A po-monoid is **integrally closed** if it satisfies

$$xy \leq y \implies x \leq 1, \qquad xy \leq x \implies y \leq 1.$$

Integral po-monoids are integrally closed, as are cancellative po-monoids.

A po-monoid is **cancellative** if it satisfies

$$xz \leq yz \implies x \leq y, \qquad zx \leq zy \implies x \leq y.$$

Integrally closed sℓ-monoids are a nuclear quasivariety but not a variety:

$$\langle [0,1], \oplus, \max \rangle \in \mathbb{H}(\langle \mathbb{R}^+, +, \max \rangle).$$

An atomic formula is **nuclear** if it has the form

$$t(x_1,\ldots,x_n) \le y.$$

A first-order formula is **nuclear** if each atomic subformula is nuclear.

Each first-order formula is equivalent to a nuclear first-order formula:

$$t(x_1,\ldots,x_n) \le u(x_1,\ldots,x_n) \equiv \forall y\,(u(x_1,\ldots,x_n) \le y \implies t(x_1,\ldots,x_n) \le y).$$

Fact. Nuclear universal sentences are preserved under $\mathbb{N}$.

Proof. For each nucleus γ on an sℓ-algebra $\mathbf{A}$ and all $a_1,\ldots,a_n,b \in \mathbf{A}_\gamma$:

$$t^{\mathbf{A}_\gamma}(a_1,\ldots,a_n) \le^{\mathbf{A}_\gamma} b \iff \gamma t(a_1,\ldots,a_n)) \le^{\mathbf{A}} b \iff t(a_1,\ldots,a_n) \le^{\mathbf{A}} b$$

THEOREM

TFAE for each universal class K of sℓ-algebras:

- K is closed under $\mathbb{N}$, i.e. it is a nuclear universal class.
- K is axiomatized by a set of **nuclear universal sentences**.
- K is axiomatized by a set of universal sentences of the form:

$$t_1 \leq x_1 \,\&\, \ldots \,\&\, t_m \leq x_m \implies u_1 \leq y_1 \text{ or } \ldots \text{ or } u_n \leq y_n . t_1 \leq x_1 \,\&\, \ldots \,\&\, t_m \leq$$

COROLLARY

TFAE for each quasivariety K of sℓ-algebras:

- K is closed under $\mathbb{N}$, i.e. it is a nuclear quasivariety.
- K is axiomatized by a set of **nuclear quasi-equations**.
- K is axiomatized by a set of quasi-equations of the form:

$$t_1 \leq x_1 \,\&\, \ldots \,\&\, t_m \leq x_m \implies u \leq y . t_1 \leq x_1 \,\&\, \ldots \,\&\, t_m \leq x_m \implies u \leq v.$$

The restriction to nuclear formulas to the right of $\implies$ is not substantial.

PROOF.

A **sub-embedding** is a map $h\colon \mathbf{A} \hookrightarrow \mathbf{B}$ such that for all $a_1, \ldots, a_n, b \in \mathbf{A}$

$$t^{\mathbf{A}}(a_1, \ldots, a_n) \leq^{\mathbf{A}} b \iff t^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) \leq^{\mathbf{B}} h(b).$$

FACT. $\mathbf{A}$ sub-embeds into $\mathbf{B}$ if and only if $\mathbf{A} \in \mathbb{INS}(\mathbf{B})$.

PROOF. Let $\mathbf{C} \leq \mathbf{B}$ be the subalgebra generated by $h[\mathbf{A}]$. Then $h[\mathbf{A}] \in \mathbb{N}(\mathbf{C})$:

Each element $c \in \mathbf{C}$ has the form

$$c = t^{\mathbf{B}}(h(a_1), \ldots, h(a_n)).$$

Take $b := t^{\mathbf{A}}(a_1, \ldots, a_n) \in \mathbf{A}$. Then for each $x \in \mathbf{A}$

$$c \leq^{\mathbf{B}} h(x) \iff t^{\mathbf{A}}(a_1, \ldots, a_n) \leq^{\mathbf{A}} x \iff b \leq^{\mathbf{A}} x \iff h(b) \leq^{\mathbf{A}} h(x).$$

So $h(b)$ is the smallest element of $h[\mathbf{A}]$ above c. This yields a nucleus on $\mathbf{C}$.

PROOF.

A **sub-embedding** is a map $h\colon \mathbf{A} \hookrightarrow \mathbf{B}$ such that for all $a_1, \dots, a_n, b \in \mathbf{A}$

$$t^{\mathbf{A}}(a_1, \dots, a_n) \leq^{\mathbf{A}} b \iff t^{\mathbf{B}}(h(a_1), \dots, h(a_n)) \leq^{\mathbf{B}} h(b).$$

FACT. **A** sub-embeds into **B** if and only if $\mathbf{A} \in \mathbb{INS}(\mathbf{B})$.

The **nuclear diagram** of an $s\ell$-algebra **A** consists of the formulas (in the signature expanded by a constant $\overline{a}$ for each $a \in \mathbf{A}$) of the forms

$$t(\overline{a}_1, \dots, \overline{a}_n) \leq \overline{b}, \qquad t(\overline{a}_1, \dots, \overline{a}_n) \nleq \overline{b},$$

which hold in **A** if we interpret $\overline{a}$ by a.

If $\langle \mathbf{B}, (\overline{a})_{a \in \mathbf{A}} \rangle$ is a model of NuclDiag(**A**), then $a \mapsto \overline{a}$ sub-embeds **A** into **B**.

If $\mathbf{A}$ validates the **nuclear universal theory** of a class of sℓ-algebras K, then the conjunction ϕ of each finite subset of NuclDiag($\mathbf{A}$) has a model in K:

$$
\begin{aligned}
\mathbf{A} \models \phi(\overline{a}_1,\dots,\overline{a}_n) &\implies \mathbf{A} \models \exists x_1 \dots x_n\, \phi(x_1,\dots,x_n) \\
&\implies \mathbf{A} \not\models \neg\exists x_1 \dots x_n\, \phi(x_1,\dots,x_n) \\
&\implies \mathbf{A} \not\models \forall x_1 \dots x_n\, \neg\phi(x_1,\dots,x_n) \\
&\implies \mathsf{K} \not\models \forall x_1 \dots x_n\, \neg\phi(x_1,\dots,x_n) \\
&\implies \mathbf{B} \models \neg\forall x_1 \dots x_n\, \neg\phi(x_1,\dots,x_n) \text{ for some } \mathbf{B} \in \mathsf{K} \\
&\implies \mathbf{B} \models \exists x_1 \dots x_n\, \phi(x_1,\dots,x_n) \text{ for some } \mathbf{B} \in \mathsf{K} \\
&\implies \mathbf{B} \models \phi(\overline{a}_1,\dots,\overline{a}_n) \text{ for some expansion of some } \mathbf{B} \in \mathsf{K}.
\end{aligned}
$$

Therefore NuclDiag($\mathbf{A}$) has a model in $\mathbb{P}_{\mathrm{U}}(\mathsf{K})$. So, $\mathbf{A} \in \mathbb{INSP}_{\mathrm{U}}(\mathsf{K})$. $\square$

Theorem

TFAE for each $s\ell$-algebra $\mathbf{A}$ and each class of $s\ell$-algebras K:

- $\mathbf{A}$ validates the nuclear universal theory of K.
- $\mathbf{A} \in \mathbb{INSP}_{\mathrm{U}}(\mathsf{K})$.

Corollary

TFAE for each $s\ell$-algebra $\mathbf{A}$ and each class of $s\ell$-algebras K:

- $\mathbf{A}$ validates the nuclear quasi-equational theory of K.
- $\mathbf{A} \in \mathbb{INSPP}_{\mathrm{U}}(\mathsf{K})$.

We get a **syntactic proof** that each integral nuclear quasivariety is a variety and each integral nuclear universal class is a positive universal class.

Consider a nuclear quasi-equation

$$t_1 \leq x_1 \ \& \ \ldots \ \& \ t_n \leq x_n \implies t(x_1, \ldots, x_m) \leq x_j.$$

Think of this as a recipe for updating the x_i's:

$$x_1' := x_1 \vee t_1 \vee \ldots, \qquad \ldots, \qquad x_n' := x_n \vee t_n \vee \ldots$$

After finitely many such updates, you get terms u_i such that the original quasi-equation is equivalent to the equation

$$t(u_1, \ldots, u_m) \leq u_j.$$

The **positive nuclear diagram** of an $s\ell$-algebra **A** consists of the formulas

$$t(\overline{a}_1, \dots, \overline{a}_n) \leq \overline{b}$$

which hold in **A** if we interpret $\overline{a}$ by a.

A **subhomomorphism** is a map $h\colon \mathbf{A} \rightharpoondown \mathbf{B}$ such that:

$$t^{\mathbf{A}}(a_1, \dots, a_n) \leq^{\mathbf{A}} b \implies t^{\mathbf{B}}(h(a_1), \dots, h(a_n)) \leq^{\mathbf{B}} h(b),$$

or equivalently

$$t^{\mathbf{B}}(h(a_1), \dots, h(a_n)) \leq^{\mathbf{B}} h(t^{\mathbf{A}}(a_1, \dots, a_n)).$$

Subhomomorphism Preservation Theorem

A first-order sentence ϕ is preserved under subhomomorphisms, i.e.

$$\mathbf{A} \vDash \phi \text{ and } h\colon \mathbf{A} \rightharpoondown \mathbf{B} \implies \mathbf{B} \vDash \phi,$$

if and only if it is equivalent to a nuclear existential positive sentence.

Nuclear and conuclear classes of residuated structures

THEOREM

TFAE for each universal class K of integral partially residuated (s)ℓ-algebras:

- K is axiomatized by universal sentences in the signature of sℓ-algebras.
- K is closed under $\mathbb{C}$.

COROLLARY

TFAE for each quasivariety K of integral partially residuated (s)ℓ-algebras:

- K is axiomatized by quasi-equations in the signature of sℓ-algebras.
- K is closed under $\mathbb{C}$.

THEOREM

TFAE for each universal class K of integral partially residuated (s)ℓ-algebras:

- K is axiomatized by positive universal sℓ-algebra sentences.
- K is closed under $\mathbb{N}$ and $\mathbb{C}$.

$\mathbb{C}$,$\mathbb{N}$-closed universal classes of integral partially residuated (s)ℓ-algebras

$\cong$

positive universal classes of integral sℓ-algebras

COROLLARY

TFAE for each quasivariety K of integral partially residuated (s)ℓ-algebras:

- K is axiomatized by sℓ-algebra equations.
- K is closed under $\mathbb{N}$ and $\mathbb{C}$.

$\mathbb{C}$,$\mathbb{N}$-closed quasivarieties of integral partially residuated (s)ℓ-algebras

$\cong$

varieties of integral sℓ-algebras

Summary

For integral $s\ell$-monoids:

- FEP for all positive universal classes.
- These are exactly the $\mathbb{N}$-closed universal classes.

For integral distributive ℓ-monoids:

- FEP for all positive universal classes.
- These are exactly the $\mathbb{N}_\wedge$-closed universal classes.

For integral residuated lattices:

- FEP for classes axiomatized by positive universal sentences in $\{\vee, \cdot, 1, \}$.
- These are exactly the $\mathbb{C}$,$\mathbb{N}$-closed positive universal classes.

For distributive integral residuated lattices:

- FEP for classes axiomatized by positive universal sentences in $\{\vee, \wedge, \cdot, 1, \}$.
- These are exactly the $\mathbb{C}_\wedge$,$\mathbb{N}_\wedge$-closed positive universal classes.

Question. Is each universal class of integral $s\ell$-monoids the class of subreducts of a universal class of integral residuated lattices?

Many other questions that one can ask about nuclear fragments.

Interpolation, definability, nuclear SAT, nuclear elementary equivalence, …

Co/nuclei merit attention at the level of categorical universal algebra.

Question. What is the role of nuclei and conuclei in the category of po-algebras and subhomomorphisms?

Thank you for your attention!