DECIDABILITY IN CLASSES OF INTEGRAL RESIDUATED LATTICES WITH A CONUCLEUS

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ABSTRACT. The notion of a conucleus provides a natural extension of topological interior operators (a.k.a. S4 modal operators) on Boolean algebras to residuated lattices. Using algebraic methods involving well-behaved embeddings of finite partial subalgebras, we establish a number of decidability results for varieties of (bounded) integral residuated lattices equipped with a conucleus. We show that the variety of Abelian ℓ-group cones equipped with a conucleus (or a meet preserving conucleus, or a join preserving conucleus) has a decidable universal theory. As a consequence, so do MV-algebras equipped with a (meet preserving, join preserving) conucleus and Abelian ℓ-groups equipped with a negative (meet preserving, join preserving) conucleus. The Montagna–Tsinakis representation of cancellative commutative residuated lattices as conuclear images of Abelian lattice-ordered groups then yields as a corollary that the variety of integral cancellative commutative residuated lattices and the subvarieties of fully distributive ones and of semilinear ones also have a decidable universal theory. This settles a long-standing open problem in the first case and recovers a result of Horčík in the last case. We also show that given a variety of integral residuated lattices with locally finite monoid reducts which enjoys the Finite Embeddability Property, this property is inherited by the variety of its conuclear expansions.

1. INTRODUCTION

Classical modal logic S4, one of the most distinguished denizens of the zoo of modal logics, is the modal logic of preorders, or equivalently the logic of the interior operator on topological spaces [16, 10]. Its algebraic semantics is given by Boolean algebras equipped with an interior operator $\Box$ satisfying the equations $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$. When we move beyond classical logic to the study of substructural logics [25, 36] (a wide family which among others includes super-intuitionistic logics, fuzzy logics, and relevance logics), Boolean algebras are replaced by *residuated lattices*: algebras of the form $\mathbf{A} := \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ with a lattice structure, a multiplicative monoidal structure, and two division operations (also called residuals) satisfying the so-called residuation law. Besides Boolean algebras, structures like Heyting algebras, MV-algebras, Sugihara monoids, and lattice-ordered groups (ℓ-groups for short) are all examples of residuated lattices. (In Boolean algebra and Heyting algebras, products and meets coincide.) What is the appropriate notion of an S4-like modal operator on residuated lattices?

From the perspective of algebraic semantics, which we confine our attention to throughout this paper, the notion of a *conucleus* has proved to be a very fruitful answer. A conucleus is an interior operator on a residuated lattice whose image is a submonoid, or equivalently an interior operator $\Box$ which satisfies the inequalities $\Box x \cdot \Box y \leq \Box(x \cdot y)$ and $1 \leq \Box 1$. Conuclei on Heyting algebras (in particular, on Boolean algebras) are precisely the interior operators which preserve finite meets.

The importance of this definition stems from the fact that conuclei provide a convenient and flexible way of constructing new residuated lattices. The image of a conucleus $\Box$ on a residuated lattice $\mathbf{A}$ always carries the structure of a residuated lattice $\mathbf{A}_\Box$, though it need not be a subalgebra of $\mathbf{A}$: joins and products in $\mathbf{A}_\Box$ coincide with those in $\mathbf{A}$ but meets and divisions in $\mathbf{A}_\Box$ are $\Box$-images of those in $\mathbf{A}$. Residuated lattices of the form $\mathbf{A}_\Box$ will be called *conuclear images* of $\mathbf{A}$.

For example, the Heyting algebra of all upsets of a poset $\langle X, \leq \rangle$ arises as a conuclear image of the Boolean algebra of all subsets of the set X, where the conucleus is the interior operator with respect to the topology of upward closed sets.

Another important instance of this construction is the *negative cone* $\mathbf{A}^-$ of a residuated lattice $\mathbf{A}$, which is $\mathbf{A}_\Box$ for the conucleus $\Box x := 1 \wedge x$. The negative cone of $\mathbf{A}$ is thus a residuated lattice over the set $\{a \in \mathbf{A} \mid a \leq 1\}$. For example, the Abelian ℓ-group of reals $\mathbb{R}$ can be seen as a residuated lattice where the monoidal operation is given by addition and the residuals by subtraction. Its negative cone is a residuated lattice $\mathbb{R}^-$ where the monoidal operation is given by addition and the residuals by truncated subtraction $x \ominus y := \min(x - y, 0)$. This is an example of an *Abelian ℓ-group cone* (the negative cone of an Abelian ℓ-group).

Conuclei and conuclear images are fundamental, either explicitly or implicitly, to the structure theory of numerous classes of residuated lattices. The conuclear image construction clearly preserves all universal sentences involving only joins and products (such as integrality, commutativity, and cancellativity), but it may fail to preserve those which involve meets or divisions (such as distributivity or involutivity). This leads to representation theorems which describe a given class of (bounded) residuated lattices as precisely the class $\mathbb{C}(\mathsf{K})$ of all conuclear images of some smaller class of (bounded) residuated lattices K. The earliest theorem of this sort is the result of McKinsey and Tarski [35, Theorem 1.15] that Heyting algebras are precisely the conuclear images of Boolean algebras, since the free Boolean extension of a Heyting algebra $\mathbf{H}$ can be expanded by a conucleus whose image is $\mathbf{H}$. We may state this result more succinctly as: $\mathbb{C}(\mathsf{BA}) = \mathsf{HA}$.

A more recent example, which will play an important role in this paper, is the result of Montagna and Tsinakis [37, Corollar 5.2] that the conuclear images of Abelian ℓ-groups are precisely the cancellative commutative residuated lattices. Further examples of representation theorems in this genre restrict to particular kinds of conuclei, such as negative cones. For instance, the negative cones of (Abelian) ℓ-groups [5, Theorem 6.2] are precisely the cancellative divisible integral (commutative) residuated lattices, and the negative cones of bounded idempotent involutive residuated lattices are precisely Heyting algebras [22, Fact 4.34].

This paper deals with the problem of deciding the validity of equations, or more generally of universal sentences, in classes of residuated lattices equipped with a conucleus. That is, given a class K of (bounded) residuated lattices, we wish to know whether the class $\mathbb{C}\mathsf{x}(\mathsf{K})$ of all expansions of K-algebras by a conucleus (*conuclear* K*-algebras* for short) has a decidable equational or universal theory. More precisely, our methods are restricted to *integral* residuated lattices (IRLs for short), where the monoidal unit 1 is also the top element of the lattice reduct.

The algebraic method of choice for proving decidability is to establish the *Finite Model Property* (FMP for short) or the *Finite Embeddability* Property (FEP for short), which state that each equation or each universal sentence which fails in a class K in fact fails in some finite K-algebra [36]. The earliest result of this sort is the proof of the FEP for the variety of Boolean algebras equipped with a conucleus (a.k.a. S4 modal algebras) due to McKinsey [34, Theorem 12], followed by the proof of the FEP for Heyting algebras due to McKinsey and Tarski [35, Theorem 1.11]. We may state this result more succinctly as: $\mathbb{C}\mathsf{x}(\mathsf{BA})$ has the FEP. We show that this result extends, in a uniform fashion, beyond the variety of Boolean algebras. For example, if K is a variety of Heyting algebras with the FEP, then the variety of conuclear K-algebras has the FEP.

Conuclear IRLs inherit the FEP (Theorem 3.10)**.** *Consider a universal class K of (bounded) IRLs with locally finite monoid reducts. If K has the FEP, then so does $\mathbb{C}\mathsf{x}(\mathsf{K})$.*

The FEP for bounded conuclear IRLs was already studied by Amano [2, 3], who showed that the whole variety has the FEP (see also [44, Theorem 13]).

The main contribution of this paper is to show that the varieties of conuclear MV-algebras and of conuclear Abelian ℓ-group cones have a decidable universal theory. This further implies that Abelian ℓ-groups equipped with a *negative conucleus* (a conucleus whose image is contained in the negative cone) also have a decidable universal theory. While conuclear Abelian ℓ-groups and conuclear Abelian ℓ-group cones do not, for trivial reasons, enjoy the FEP (there are no non-trivial finite Abelian ℓ-groups and ℓ-group cones), conuclear MV-algebras do. The decidability result for conuclear Abelian ℓ-group cones may be proved directly, but it may also be derived as a consequence of the FEP for conuclear MV-algebras.

We further prove analogous results for *meet preserving* conuclei ($\wedge$-*conuclei* for short) and *join preserving conuclei* ($\vee$-*conuclei* for short), defined as those conuclei which satisfy $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box(x \vee y) = \Box x \vee \Box y$, respectively. The notation $\mathbb{C}_\wedge$, $\mathbb{C}\mathsf{x}_\wedge$ and $\mathbb{C}_\vee$, $\mathbb{C}\mathsf{x}_\vee$ is self-explanatory. For $\wedge$-conuclei we obtain entirely analogous results, for $\vee$-conuclei the only difference is that while the variety of $\vee$-conuclear MV-algebras has a decidable universal theory, it does not have the FEP.

Although we shall not explicitly develop this logical angle here, each of our decidability results is equivalent to the decidability of the deducibility problem (the problem of deciding whether a sequence of propositional formulas implies a propositional formula) for a certain non-classical modal logic, namely for certain variants of S4 Łukasiewicz logic and S4 modal Abelian logic. Just like S4 is one of the most important classical modal logics, Łukasiewicz logic (which replaces the classical two-element set of truth values $\{0, 1\}$ by the real unit interval $[0, 1]$ with a suitable MV-algebraic structure) is among the best-studied non-classical logics [17, 39]. Studying S4-like modal logics over a Łukasiewicz non-modal basis is therefore extremely natural. For comparison with other existing decidability and undecidability results for many-valued modal logics, the reader may consult [46].

Decidability results for conuclear residuated lattices (Theorems 5.10, 7.22, 5.15, 7.24, 6.10, 7.31, 8.4)**.** *The following varieties have a decidable universal theory:*

- *(i) Abelian ℓ-group cones with a conucleus,*
- *(ii) Abelian ℓ-group cones with a $\wedge$-conucleus,*
- *(iii) Abelian ℓ-group cones with a $\vee$-conucleus,*
- *(iv) Abelian ℓ-groups with a negative conucleus,*
- *(v) Abelian ℓ-groups with a negative $\wedge$-conucleus,*
- *(vi) Abelian ℓ-groups with a negative $\vee$-conucleus,*
- *(vii) MV-algebras with a conucleus.*
- *(viii) MV-algebras with a $\wedge$-conucleus.*

(ix) *MV-algebras with a $\vee$-conucleus.*

The Finite Embeddability Property for conuclear MV-algebras (Theorem 6.8, Theorem 7.30, Corollary 8.6)**.**

(i) *The variety of MV-algebras with a conucleus has the FEP.*
(ii) *The variety of MV-algebras with a $\wedge$-conucleus has the FEP.*
(iii) *The variety of MV-algebras with a $\vee$-conucleus does not have the FEP.*

We are now in a position to derive, as a corollary, the decidability results which in fact formed the original impetus for the present work. Namely, it is straightforward to observe that for any class of residuated lattices K, the decidability of the universal theory of $\mathbb{C}\mathsf{x}(\mathsf{K})$ implies the decidability of the universal theory of $\mathbb{C}(\mathsf{K})$, and likewise for $\mathbb{C}\mathsf{x}_{\wedge}$, $\mathbb{C}_{\wedge}$ and for $\mathbb{C}\mathsf{x}_{\vee}$, $\mathbb{C}_{\vee}$.

The classes of conuclear, $\wedge$-conuclear, and $\vee$-conuclear images of Abelian ℓ-groups and therefore also of Abelian ℓ-group cones were described by Montagna and Tsinakis [37] in their seminal study of conuclei on ℓ-groups. They in fact established a number of categorical equivalences between classes of conuclear ℓ-groups and classes of cancellative residuated lattices, but we shall not need the full strength of these equivalences here.

Let us now introduce the terminology required to describe these classes of conuclear images. We denote the variety of integral commutative ($x \cdot y = y \cdot x$) residuated lattices by ICRL. An ICRL is *fully distributive* if its satisfies the equations

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z).$$

An ICRL is called *semilinear* if it is isomorphic to a subdirect product of totally ordered ICRLs, or equivalently if it satisfies the equation $(x \backslash y) \vee (y \backslash x) = 1$. A cancellative ICRLs is semilinear if and only if it satisfies $x \backslash (y \vee z) = (x \backslash y) \vee (x \backslash z)$. Recall that AbLG^{-} denotes the variety of Abelian ℓ-group cones.

The Montagna–Tsinakis representation ([37])**.**

(i) $\mathbb{C}(\mathsf{AbLG}^{-})$ *is the variety of cancellative* ICRL*s.*
(ii) $\mathbb{C}_{\wedge}(\mathsf{AbLG}^{-})$ *is the variety of fully distributive cancellative* ICRL*s.*
(iii) $\mathbb{C}_{\vee}(\mathsf{AbLG}^{-})$ *is the variety of semilinear cancellative* ICRL*s.*

Given the Montagna–Tsinakis representation, the above decidability results immediately yield the following corollaries.

Decidability results for cancellative residuated lattices (Theorems 5.18, 7.27, 8.4)**.** *The following varieties have a decidable universal theory:*

(i) *cancellative* ICRL*s.*
(ii) *fully distributive cancellative* ICRL*s.*
(iii) *semilinear cancellative* ICRL*s* [29].

This settles the long-standing problem of whether the variety of cancellative ICRLs has a decidable equational theory [36, Problem 26], which had remained open since the first systematic study of cancellative residuated lattices in [5]. This problem had not seen substantial progress since Horčík [29] settled the semilinear case in 2006.[1] All of the other decidability results above besides Horčík's are new.

[1]Horčík [29] does not explicitly discuss universal theories, but for a quasivariety the decidability of the quasi-equational theory is equivalent to the decidability of the universal theory. What he in effect proves is that the universal class of totally ordered cancellative ICRLs has a decidable universal theory.

1.1. **Summary.** Now that we have laid out our main results, in the remainder of this introductory section we provide a bird's eye view of the whole paper.

All of the universal classes which feature in our decidability results are finitely axiomatizable, so their universal theories are recursively enumerable, given the existence of a proof calculus for universal sentences [42]. It remains to show that they are also co-recursively enumerable by finding suitable generating classes.

Accordingly, the main problem that we need to tackle in this paper is the following, where IRL denotes the variety of integral residuated lattices:

$$\text{given a variety } \mathsf{V} \subseteq \mathsf{IRL}, \text{ find a useful generating class for } \mathbb{C}\mathrm{x}(\mathsf{V}).$$

We shall be concerned exclusively with finding a class that generates $\mathbb{C}\mathrm{x}(\mathsf{V})$ as a *universal class*: recall that the universal class generated by a class of algebras K in a given signature is $\mathbb{U}(\mathsf{K}) := \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$. Moreover, our strategy will be to reduce this problem to one which is stated purely in the language of residuated lattices (rather than conuclear residuated lattices). The problem thus becomes:

$$\text{given a variety } \mathsf{V} \subseteq \mathsf{IRL}, \text{ find a useful class } \mathsf{K} \subseteq \mathsf{V} \text{ such that } \mathbb{C}\mathrm{x}(\mathsf{V}) = \mathbb{UC}\mathrm{x}(\mathsf{K}).$$

The bulk of the present paper will be spent on trying to solve this problem. The decidability results listed above then follow as fairly straightforward corollaries of such generation results.

The key to the problem lies in identifying the right analogy between the class operators $\mathbb{U}$ and $\mathbb{UC}\mathrm{x}\mathbb{S}$. A classical result states that given an algebra $\mathbf{A}$ and a class of algebras K in the same signature, $\mathbf{A} \in \mathbb{U}(\mathsf{K})$ if and only if each finite partial subalgebra of $\mathbf{A}$ has an embedding into K. This means that for each finite set $X \subseteq \mathbf{A}$ there is an injective map $\iota\colon X \hookrightarrow \mathbf{B}$ into some $\mathbf{B} \in \mathsf{K}$ such that for each n-ary operation $\circ$ in the signature and for all $a_1, \dots, a_n, b \in X$

$$\circ^{\mathbf{A}}(a_1, \dots, a_n) = b \implies \circ^{\mathbf{B}}(\iota(a_1), \dots, \iota(a_n)) = \iota(b).$$

The entire paper now hinges on the following simple lemma, which replaces embeddings by what we call *π-embeddings*. These are embeddings which further require that for all $a_1, \dots, a_n, b \in X$

$$a_1 \cdot \dots \cdot a_n \leq^{\mathbf{A}} b \iff \iota(a_1) \cdot \dots \cdot \iota(a_n) \leq^{\mathbf{B}} \iota(b).$$

$\mathbb{UC}\mathrm{x}\mathbb{S}$ **via π-embeddings** (Lemma 3.7). *If each finite partial subalgebra of $\mathbf{A} \in \mathsf{IRL}$ has a π-embedding into $\mathsf{K} \subseteq \mathsf{IRL}$, then $\mathbb{C}\mathrm{x}\mathbb{S}(\mathbf{A}) \subseteq \mathbb{UC}\mathrm{x}\mathbb{S}(\mathsf{K})$.*

The above lemma encapsulates all that we need to know about conuclei to prove the desired generation result for conuclear Abelian ℓ-group cones. The proof will work directly with π-embeddings rather than with conuclei.

A common method of proving that a finitely axiomatizable universal class K has a decidable universal theory is to establish the *Finite Embeddability Property* or the *FEP* for short. This property states that each universal sentence which fails in K in fact fails in some finite K-algebra. If K is a universal class, this is equivalent to saying that $\mathsf{K} = \mathbb{U}(\mathsf{K}_{\mathrm{fin}})$, where $\mathsf{K}_{\mathrm{fin}}$ is the subclass of finite K-algebras.

As a consequence of the above lemma on π-embeddings, $\mathbb{C}\mathrm{x}(\mathsf{K})$ and $\mathbb{C}(\mathsf{K})$ sometimes inherit the FEP from K. This occurs for example when K is an n-potent subvariety of ICRL with the FEP.

Moreover, his methods would have sufficed to prove the stronger claim that the universal class of totally ordered conuclear Abelian ℓ-group cones have a decidable universal theory, but unfortunately the use of conuclei remains only implicit in [29].

FEP for conuclear IRL**s** (Theorem 3.10)**.** *Consider a universal class* $\mathsf{K} \subseteq \mathsf{IRL}$ *with locally finite monoid reducts. If* K *has the FEP, then so do* $\mathbb{C}\mathsf{x}(\mathsf{K})$ *and* $\mathbb{C}(\mathsf{K})$*.*

The above result unifies a number of results which have previously been proved on a case by case basis. For example, the variety of Boolean algebras clearly has the FEP by virtue of being locally finite. By the above theorem, the variety of S4 modal algebras and the variety of Heyting algebras then inherit the FEP, and the variety of conuclear Heyting algebras in turn inherits the FEP from Heyting algebras. We thereby get in one fell swoop the classical results of McKinsey [34] (for S4 modal algebras) and McKinsey and Tarski [35] (for Heyting algebras), as well as the more recent result of de Groot and Shillito [21] (conuclear Heyting algebras). We may also observe that the variety of conuclear Gödel algebras inherits the FEP from the locally finite variety of Gödel algebras.

In the more special case where K-algebras themselves are locally finite, we may also show that the class mK defined in [18] has the FEP (Theorem 3.6). This class is related to the class of monadic K-algebras and thereby to one-variable fragments of first-order logics, though it does necessarily coincide with monadic K-algebras as normally understood (see [18] for more detail).

The observant reader will have noticed that in fact Heyting algebras and Gödel algebras are not varieties of residuated lattices, but rather varieties of *bounded* residuated lattices. While we state our results for integral residuated lattices, they hold equally well for bounded integral residuated lattices, with identical or even simpler proofs. Since at no point in the paper does the presence of a bottom constant create any complications, we take the liberty of omitting explicit formulations of the obvious bounded variants of our results.

We now set our sights on our main results. Its proof consists of three parts. The first two of these, handled in Section 4, apply to the variety ICRL of integral commutative residuated lattices in general. The first part shows that we may replace π-embeddings by a broader class of maps which we call *ω-embeddings*, namely embeddings $\iota\colon X \hookrightarrow \mathbf{B}$ which require that for all $a, b, c \in X$

$$a \cdot b^n \nleq^{\mathbf{A}} c \text{ for all } n \in \mathbb{N} \implies \iota(a) \cdot \iota(b)^n \nleq^{\mathbf{B}} \iota(c) \text{ for all } n \in \mathbb{N}.$$

In fact, it suffices to work with an even broader class of maps which we call *weak ω-embeddings*. These merely require that for all $b, c \in X$

$$b^n \nleq^{\mathbf{A}} c \text{ for all } n \in \mathbb{N} \implies \iota(b)^n \nleq^{\mathbf{B}} \iota(c) \text{ for all } n \in \mathbb{N}.$$

The proof of the following theorem ultimately relies on Higman's Lemma [28].

$\mathbb{UC}\mathsf{x}\mathbb{S}$ **via (weak)** ω**-embeddings** (Theorem 4.5)**.** *If each finite partial subalgebra of* $\mathbf{A} \in \mathsf{ICRL}$ *has a (weak) ω-embedding into* $\mathsf{K} \subseteq \mathsf{ICRL}$*, then* $\mathbb{C}\mathsf{x}\mathbb{S}(\mathbf{A}) \subseteq \mathbb{UC}\mathsf{x}\mathbb{S}(\mathsf{K})$*.*

Next, we reduce the problem of finding ω-embeddings of arbitrary algebras from a variety $\mathsf{V} \subseteq \mathsf{ICRL}$ into K to the problem of finding ω-embeddings of algebras from $\mathsf{V}_{\mathrm{fsi}}$ (the class of finitely subdirectly irreducible algebras in V) into K. We use $\mathbb{P}_{\mathrm{fin}}(\mathsf{K})$ to be denote the class of all products of finite families of K-algebras.

Reduction to $\mathsf{V}_{\mathrm{fsi}}$ (Theorems 4.9 and 4.11)**.** *Consider a variety* $\mathsf{V} \subseteq \mathsf{ICRL}$*. If each finite partial subalgebra of each* $\mathbf{A} \in \mathsf{V}_{\mathrm{fsi}}$ *has a (weak) ω-embedding into a class* $\mathsf{K} \subseteq \mathsf{ICRL}$*, then*

$$\mathbb{C}\mathsf{x}(\mathsf{V}) = \mathbb{UC}\mathsf{x}\mathbb{SP}_{\mathrm{fin}}(\mathsf{K}).$$

Thus for each variety $\mathsf{V} \subseteq \mathsf{ICRL}$

$$\mathbb{C}\mathrm{x}(\mathsf{V}) = \mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}\mathbb{P}_{\mathrm{fin}}(\mathsf{V}_{\mathrm{fsi}}).$$

The final part of the proof, to which we devote Section 5, consists in applying the above reduction to the case where V is the variety of Abelian ℓ-group cones. Then $\mathsf{V}_{\mathrm{fsi}}$ is the class of totally ordered Abelian ℓ-group cones. It remains to determine the target class K and show that each finite partial subalgebra of each totally ordered Abelian ℓ-group cone indeed has a weak ω-embedding into K.

This step relies on the Hahn representation of totally ordered Abelian groups. Given a chain Γ and a totally ordered Abelian group $\mathbf{G}$, let $\mathrm{Lex}(\Gamma, \mathbf{G})$ be the lexicographic power of $\mathbf{G}$ determined by Γ (see Section 5 for a proper definition). In particular, $\mathrm{Lex}(n, \mathbf{G})$ denotes the n-th lexicographic power of $\mathbf{G}$. The totally ordered additive groups of reals and of integers are denoted by $\mathbb{R}$ and $\mathbb{Z}$, respectively.

The Hahn representation ([27, 4])**.** *Each totally ordered Abelian group embeds into* $\mathrm{Lex}(\Gamma, \mathbb{R})$ *for some chain* Γ*.*

Armed with the Hahn representation, it is not difficult to show that each finite partial subalgebra of each totally ordered Abelian ℓ-group cone has a weak ω-embedding into $\mathrm{Lex}(n, \mathbb{Z})$ for some $n \in \mathbb{N}$. This concludes the proof of the main generation result. Deriving the decidability of the universal theory of conuclear Abelian ℓ-group cone from the generation result is a fairly simple matter.

Throughout the next sequence of results, let AbLG denote the variety of Abelian ℓ-groups, AbLG^- the variety of Abelian ℓ-group cones, and MV the variety of MV-algebras. More generally, K^- denotes the class of negative cones of algebras from K. The class operator $\mathbb{C}\mathrm{x}^-$ ($\mathbb{C}\mathrm{x}^-_\wedge$) takes all expansions of a given class of residuated lattices by a negative conucleus ($\wedge$-conucleus).

Generation results involving conuclei (Theorems 5.9, 5.14, 5.17)**.**

(i) $\mathbb{C}\mathrm{x}(\mathsf{AbLG}^-)$ *is generated as a universal class by* $\mathbb{C}\mathrm{x}\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-)$*.*
(ii) $\mathbb{C}\mathrm{x}^-(\mathsf{AbLG})$ *is generated as a universal class by* $\mathbb{C}\mathrm{x}^-\mathbb{P}_{\mathrm{fin}}(\mathbb{Z})$*.*
(iii) $\mathbb{C}(\mathsf{AbLG}^-)$ *is generated as a universal class by* $\mathbb{C}\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-)$*.*

(The class $\mathbb{C}(\mathsf{AbLG}^-)$ *is the variety of cancellative* ICRL*s* [37]*.)*

Going through the proof of the above theorem and replacing the π-embeddings by a more restrictive class of embeddings which we call $\wedge\pi$-embeddings yields an analogous result for $\wedge$-conuclei. This is the content of Section 7.

Generation results involving $\wedge$-conuclei (Theorems 7.20, 7.23, 7.26)**.**

(i) $\mathbb{C}\mathrm{x}_\wedge(\mathsf{AbLG}^-)$ *is generated as a universal class by* $\mathbb{C}\mathrm{x}_\wedge\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-)$*.*
(ii) $\mathbb{C}\mathrm{x}^-_\wedge(\mathsf{AbLG})$ *is generated as a universal class by* $\mathbb{C}\mathrm{x}^-_\wedge\mathbb{P}_{\mathrm{fin}}(\mathbb{Z})$*.*
(iii) $\mathbb{C}_\wedge(\mathsf{AbLG}^-)$ *is generated as a universal class by* $\mathbb{C}_\wedge\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-)$*.*

(The class $\mathbb{C}_\wedge(\mathsf{AbLG}^-)$ *is the variety of fully distributive cancellative* ICRL*s* [37]*.)*

We also derive the following results for $\vee$-conuclei in Section 8, where

$$\mathsf{Lex} := \{\mathrm{Lex}(n, \mathbb{Z}) \mid n \in \mathbb{N}\}.$$

Generation results involving $\vee$-conuclei (Theorem 8.4)**.**

(i) $\mathbb{C}\mathrm{x}_\vee(\mathsf{AbLG}^-)$ *is generated as a quasivariety by* $\mathbb{C}\mathrm{x}_\vee\mathbb{S}(\mathsf{Lex}^-)$*.*
(ii) $\mathbb{C}\mathrm{x}^-_\vee(\mathsf{AbLG})$ *is generated as a quasivariety by* $\mathbb{C}\mathrm{x}^-_\vee\mathbb{S}\mathbb{P}_{\mathrm{fin}}(\mathsf{Lex})$*.*
(iii) $\mathbb{C}_\vee(\mathsf{AbLG}^-)$ *is generated as a quasivariety by* $\mathbb{C}_\vee\mathbb{S}\mathbb{P}_{\mathrm{fin}}(\mathsf{Lex}^-)$*.*

(The class $\mathbb{C}_{\vee}(\mathsf{AbLG}^{-})$ *is the variety of semilinear cancellative* ICRLs [37].*)*

Analogous results for MV-algebras, to which we devote Section 6, rely on the unit interval construction. The *unit interval* of an integral residuated lattice $\mathbf{A}$ induced by $u \in \mathbf{A}$ is a bounded integral residuated lattice over the interval $[u, 1]_{\mathbf{A}}$ of $\mathbf{A}$ which inherits the lattice structure and division from $\mathbf{A}$ and but replaces the multiplication of $\mathbf{A}$ by $x \circ y := u \vee (x \cdot y)$. The unit intervals of a *conuclear* integral residuated lattice with a conucleus $\Box$ are the unit intervals $[u, 1]_{\mathbf{A}}$ for $u \in \mathbf{A}_{\Box}$ equipped with the restriction of the conucleus $\Box$ to $[u, 1]_{\mathbf{A}}$.

To derive generation results for conuclear MV-algebras from generation results for Abelian ℓ-group cones, it will suffice to apply the classical representation result of Mundici [38] together with an unpublished result of Young [48] which states that $\mathbb{C}\mathsf{x}$ and $\mathbb{I}\mathsf{nt}$ commute in the context of distributive integral residuated lattices. The difficult part of Young's result is showing that a conucleus on an interval of a distributive integral residuated lattice $\mathbf{A}$ extends to a conucleus on $\mathbf{A}$. Given a class K of (conuclear) integral residuated lattices, we denote the class of all unit intervals of K-algebras by $\mathbb{I}\mathsf{nt}(\mathsf{K})$ in the following.

The Mundici representation ([38])**.** *MV-algebras are precisely the unit intervals of Abelian ℓ-group cones:* $\mathsf{MV} = \mathbb{I}\mathsf{nt}(\mathsf{AbLG}^{-})$.

The Young representation ([48])**.** *Conuclear MV-algebras are precisely the unit intervals of conuclear Abelian ℓ-group cones:* $\mathbb{C}\mathsf{x}(\mathsf{MV}) = \mathbb{I}\mathsf{nt}(\mathbb{C}\mathsf{x}(\mathsf{AbLG}^{-}))$.

The class of finite MV-algebras will be denoted by $\mathsf{MV}_{\mathsf{fin}}$.

Generation results involving MV-algebras (Theorems 6.8, 7.30, 8.4)**.**

(i) $\mathbb{C}\mathsf{x}(\mathsf{MV})$ *is generated as a universal class by* $\mathbb{C}\mathsf{x}(\mathsf{MV}_{\mathsf{fin}})$*, i.e. it has the FEP.*
(ii) $\mathbb{C}\mathsf{x}_{\wedge}(\mathsf{MV})$ *is generated as a universal class by* $\mathbb{C}\mathsf{x}_{\wedge}(\mathsf{MV}_{\mathsf{fin}})$*, i.e. it has the FEP.*
(iii) $\mathbb{C}\mathsf{x}_{\vee}(\mathsf{MV})$ *is generated as a quasivariety by* $\mathbb{C}\mathsf{x}\mathbb{S}(\mathbb{I}\mathsf{nt}(\mathsf{Lex}^{-}))$.

The class $\mathbb{C}\mathsf{x}_{\vee}(\mathsf{MV})$ does not have the FEP (Corollary 8.6).

1.2. **Open problems.** While the results of this paper constitute a significant advance in our understanding of cancellative residuated lattices, conuclei on Abelian ℓ-groups, and conuclear integral residuated lattices in general, they are also constrained by several limitations. A number of further directions in which these results may be developed further therefore naturally offer themselves.

The first direction is to obtain uniform FEP theorems for varieties of conuclear integral residuated lattices beyond the case of locally finite monoid reducts. We know that the methodology of Blok and van Alten [11] applies to conuclear IRLs, since it was used by Amano [2, 3] to prove that the variety of all bounded conuclear IRLs has the FEP (see also [44, Theorem 13]). However, it remains to develop this direction further. In particular, a major result for IRLs due to Galatos and Jipsen [24, Theorem 3.18] states that all varieties of IRLs axiomatized by equations in the signature $\{\vee, \cdot, 1\}$ have the FEP. The proof uses the technology of residuated frames developed in [24].

Open Problem 1. *Adapt the technology of residuated frames to get uniform FEP results beyond the case of locally finite monoid reducts. Does each variety of conuclear integral residuated lattices axiomatized by equations in the signature* $\{\vee, \cdot, 1\}$ *satisfy the FEP?*

A second direction is to determine the precise computational complexity of the equational and universal theories whose decidability we prove in this paper. Looking at their lattice reducts suffices to determine that all of these theories are co-NP-hard: the lattice fragment of the equational theory of cancellative ICRLs is the equational theory of lattices [41, Corollary 4.3], while the lattice of fragment of the equational theories of all the other varieties involved in our decidability results is the equational theory of distributive lattices. Both of these equational theories are known to be co-NP-complete (see [43, 19, 12] and [30]).

The universal theory of Abelian ℓ-groups is also co-NP-complete [47], and consequently the same holds for the universal theories of Abelian ℓ-group cones and of MV-algebras (by Lemma 2.7 and its analogue for nuclear images). On the other hand, the equational theory of conuclear Boolean algebras (S4 modal algebras) is PSPACE-complete [31]. Beyond the Abelian case, the universal theory of ℓ-groups is known to be undecidable [26].

Open Problem 2. *What is the computational complexity of the equational or the universal theory of Abelian ℓ-group cones with a (meet preserving, or join preserving) conucleus? What is the computational complexity of the equational or the universal theory of (fully distributive, or semilinear) integral cancellative commutative residuated lattices?*

A third direction is to go beyond the integral case, which constitutes the most substantial limitation of the methodology developed in this paper. While the Montagna–Tsinakis representation extends beyond the integral case, our methods do not. The following questions thus remain unsolved.

Open Problem 3. *Do the following classes have a decidable universal theory?*

- *(i) Abelian ℓ-groups with a conucleus ($\wedge$-conucleus, $\vee$-conucleus),*
- *(ii) (fully distributive, semilinear) cancellative commutative residuated lattices,*
- *(iii) totally ordered Abelian ℓ-groups with a conucleus,*
- *(iv) totally ordered cancellative commutative residuated lattices.*

Finally, the decidability of the universal theories of $\mathbb{C}\mathsf{x}(\mathsf{MV})$, $\mathbb{C}\mathsf{x}_\wedge(\mathsf{MV})$, and $\mathbb{C}\mathsf{x}_\vee(\mathsf{MV})$ implies that $\mathbb{C}(\mathsf{MV})$, $\mathbb{C}_\wedge(\mathsf{MV})$, and $\mathbb{C}_\vee(\mathsf{MV})$ also have decidable universal theories. However, we lack an intrinsic description of these classes. The problem of describing $\mathbb{C}(\mathsf{MV})$ was already posed in [36, Problem 10].

Open Problem 4. *Describe the classes $\mathbb{C}(\mathsf{MV})$, $\mathbb{C}_\wedge(\mathsf{MV})$, and $\mathbb{C}_\vee(\mathsf{MV})$.*

A more modest question also arises naturally in connection with our results. Given a variety $\mathsf{V} \subseteq \mathsf{ICRL}$, we suspect that in general the class $\mathbb{C}(\mathsf{V})$ of all conuclear images of V-algebras need not be a variety or even a quasivariety. However, we do not have an explicit counter-example.

Open Problem 5. *Find a variety $\mathsf{V} \subseteq \mathsf{ICRL}$ such that $\mathbb{C}(\mathsf{V})$ is not closed under $\mathbb{S}$ (under $\mathbb{H}$), or prove that no such variety exists. Do the same for $\mathbb{C}_\wedge$ and $\mathbb{C}_\vee$.*

2. Preliminaries

2.1. Order theory.

Given a subset X of some poset P, we use the notation $\downarrow X$ for the *downset* and the notation $\uparrow X$ for the *upset* generated by X. That is,

$$\downarrow X := \{p \in P \mid p \leq x \text{ for some } x \in X\},$$
$$\uparrow X := \{p \in P \mid p \geq a \text{ for some } x \in X\},$$

with $\downarrow x := \downarrow\{x\}$ and $\uparrow x := \uparrow\{x\}$ for $x \in P$. An upset is *principal* if it has the form $\uparrow x$ for some $x \in P$. It is *finitely generated* if it has the form $\uparrow X$ for some finite set X, or equivalently if it is a finite union of principal upsets. The upsets of P ordered by inclusion form a bounded distributive lattice $\mathrm{Up}\, P$.

The set X is *coinitial* if for each $p \in P$ it intersects $\downarrow p$, i.e. if for each $p \in P$ there is some $x \in X$ with $x \leq p$. The set of *maximal elements* of X will be denoted by $\max X$. A *lower bound* of X is some $p \in P$ such that $p \leq x$ for each $x \in X$.

An *interior operator* on a poset P is an order-preserving map $\Box\colon P \to P$ such that $\Box\Box x = \Box x \leq x$ for each $x \in P$. A *closure operator* on a poset P is an order-preserving map $\Diamond\colon P \to P$ such that $\Diamond\Diamond x = \Diamond x \geq x$ for each $x \in P$.

A *totally ordered set* or a *chain* is a poset where for all x and y either $x \leq y$ or $y \leq x$. A *discretely ordered set* or an *antichain* is a poset where $x \leq y$ implies $x = y$ for all x and y. In other words, an antichain is a set where no two distinct elements are comparable.

2.2. **Universal algebra.** We now review some basic terminology and notation of universal algebra. The reader may consult [13, 6] for more detail.

We use $\mathbb{H}$, $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathrm{fin}}$, $\mathbb{P}_{\mathrm{U}}$ to denote closure of a class of algebras under homomorphic images, isomorphic images, products, finite products, and ultraproducts. Here and throughout the paper, by a class of algebras we always mean a class in some common algebraic signature consisting of finitary function symbols. Algebras which belong to a class K will also be called K-*algebras* for short. The subalgebra of an algebra $\mathbf{A}$ generated by a set $X \subseteq \mathbf{A}$ will be denoted by $\mathrm{Sg}^{\mathbf{A}}\, X$.

A *(quasi)equational class* or *(quasi)variety* is a class of algebras axiomatized by a set of (quasi)equations, while a *universal class* is a class of algebras axiomatized by a set of universal sentences. The universal class generated by a class of algebras K is $\mathbb{U}(\mathsf{K}) := \mathbb{I}\mathbb{S}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$, while the quasiequational class generated by a class of algebras K is $\mathbb{Q}(\mathsf{K}) := \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$. Given a variety V, its class of finitely subdirectly irreducible algebras will be denoted by $\mathsf{V}_{\mathrm{fsi}}$. Each variety V satisfies $\mathsf{V} = \mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{V}_{\mathrm{fsi}})$.

Lemma 2.1. *A quasivariety has a decidable universal theory if and only if it has a decidable quasi-equational theory.*

Proof. A class closed under finite products satisfies a universal sentence ϕ if and only if it satisfies at least one of its finitely many "sub-quasi-equations". □

A *partial algebra* $\mathbf{A}$ is a set A equipped for each n-ary function symbol $\circ$ (in a given signature) with a partial n-ary operation $\circ^{\mathbf{A}}$ on A, i.e. a function $\circ^{\mathbf{A}}\colon X \to A$ for some $X \subseteq A^n$. An *embedding* of partial algebras $\iota\colon \mathbf{A} \hookrightarrow \mathbf{B}$ is an injective map $\iota\colon A \hookrightarrow B$ between their universes such that for each n-ary function symbol $\circ$

$$\circ^{\mathbf{A}}(a_1, \dots, a_n) = a \text{ for } a, a_1, \dots, a_n \in A \implies \circ^{\mathbf{B}}(\iota(a_1), \dots, \iota(a_n)) = \iota(a).$$

An embedding into a class of partial algebras K is an embedding into some $\mathbf{B} \in \mathsf{K}$.

Given a (partial) algebra $\mathbf{A}$ and a set $X \subseteq A$, the *restriction of* $\mathbf{A}$ to X is the partial algebra $\mathbf{A}|_X$ with universe X whose operations are the restrictions of the operations of $\mathbf{A}$. That is, $\circ^{\mathbf{A}|_X}(a_1, \dots, a_n) := b$ for $a_1, \dots, a_n, b \in X$ if and only if $\circ^{\mathbf{A}}(a_1, \dots, a_n) = b$, otherwise $\circ^{\mathbf{A}|_X}(a_1, \dots, a_n)$ is undefined in $\mathbf{A}|_X$. Algebras of the form $\mathbf{A}|_X$ will be called *partial subalgebras* of $\mathbf{A}$.

Theorem 2.2 (Universal classes and partial embeddings [32, §8.3, Theorem 2])**.**
Let $\mathbf{A}$ *be an algebra and* K *be a class of algebras. Then* $\mathbf{A} \in \mathbb{U}(\mathsf{K})$ *if and only if each finite partial subalgebra of* $\mathbf{A}$ *embeds into* K*.*

We say that a class of algebras K has the *finite embeddability property*, or the *FEP* for short, if it is included in the universal class generated by its finite members, or equivalently if each finite partial subalgebra of each algebra $\mathbf{A} \in \mathsf{K}$ embeds into some finite algebra in K. In particular, a universal class of algebras K has the FEP if it is generated as a universal class by its finite members.

The FEP provides a useful method for proving the decidability of the universal theory of a class of algebras in a finite algebraic signature. See [36, Chapter 9] for a more detailed discussion of the FEP.

Theorem 2.3 (Decidability via the FEP)**.**
Let K *be a universal class in a finite algebraic signature. If* K *is finitely axiomatizable and has the FEP, then the universal theory of* K *is decidable.*

Proof. The finiteness of the algebraic signature ensures that the notion of the universal theory being decidable is unproblematic. In particular, the universal theory of K is decidable if and only if this theory and its complement (in the set of all universal sentences in the signature of K) are both recursively enumerable.

Suppose that K is axiomatized by some finite set of universal sentences Σ. The universal theory of K is recursively enumerable by the completeness theorem of Quackenbush [42]: if a universal sentence is valid in K, then it has a proof from the finite set Σ in a certain proof calculus. On the other hand, the complement of the universal theory of K is also recursively enumerable by the FEP: a universal sentence is not valid in K if and only if it fails in some valuation on some finite algebra satisfying Σ. But we may enumerate all finite algebras which satisfy the finite set Σ and all valuations on these algebras. □

A variety of algebras V has the *finite model property* or the *FMP* for short if it is generated by its finite members. For varieties which have *equationally definable principal congruences* or *EDPC* for short, the FMP and the FEP coincide.[2]

Theorem 2.4 (FMP + EDPC = FEP + EDPC [11, Theorem 3.3])**.**
Let V *be variety of algebras with EDPC. Then* V *has the FEP if and only if* V *has the FMP.*

2.3. **Residuated lattices.** We review the basic classes of ordered and residuated structures which we will encounter in this paper. For more details, the reader is advised to consult the monographs [25, 36].

A *(commutative) partially ordered monoid* or *pomonoid* for short is an ordered algebra $\langle A, \leq, \cdot, 1\rangle$ such that $\langle A, \leq\rangle$ is a poset, $\langle A, \cdot, 1\rangle$ is a (commutative) monoid, and multiplication is order preserving in both arguments. A pomonoid is called *integral* if 1 is its top (largest) element.

A *(commutative) sℓ-monoid* is an algebra $\langle A, \vee, \cdot, 1\rangle$ such that $\langle A, \vee\rangle$ is a join semilattice, $\langle A, \cdot, 1\rangle$ is a (commutative) monoid, and the following equations hold:

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot y), \qquad (x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z).$$

---

[2] In [11, Theorem 3.3] this result is proved for quasivarieties in a finite algebraic signature, but this restriction is not necessary. The restriction is inherited from [11, Theorem 3.1], but that theorem in fact holds for arbitrary signatures: given a universal sentence ϕ, if each of its finitely many "sub-quasi-equations" ϕ_i for $i \in I$ fails in a finite algebra $\mathbf{A}_i$, then ϕ fails in the finite algebra $\prod_{i \in I} \mathbf{A}_i$.

Each sℓ-monoid has a pomonoid reduct with $x \leq y \iff x \vee y = y$.

A *(commutative) residuated lattice* is an algebra of the form $\langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that $\langle A, \wedge, \vee \rangle$ is a lattice $\langle A, \cdot, 1 \rangle$ is a (commutative) monoid, and the binary *division* operations $\backslash$ and $/$, also called the *residuals* of multiplication, satisfy the following *residuation law*, where $\leq$ denotes the order induced by the lattice $\langle A, \wedge, \vee \rangle$:

$$x \leq z/y \iff x \cdot y \leq z \iff y \leq x \backslash z.$$

The two residuals coincide ($x \backslash y = y/x$) if and only if the residuated lattice is commutative. Each residuated lattice has an sℓ-monoid reduct $\langle A, \vee, \cdot, 1 \rangle$.

The varieties of integral, of commutative, and of integral commutative residuated lattices will be denoted by IRL, CRL, and ICRL, respectively.

A *bounded residuated lattice* is a residuated lattice equipped with constants $\top$ and $\bot$ such that $\top$ is the top (largest) element and $\bot$ is the bottom (smallest) element. Equivalently, we may treat $\top$ as an abbreviation for $\bot \backslash \bot = \bot / \bot$. In an integral bounded residuated lattice $\top = 1$.

A residuated lattice is *idempotent* if it satisfies the equation $x^2 = x$. More generally, it is called *n-potent* for $n \in \mathbb{N}$ if it satisfies the equation $x^{n+1} = x^n$. Idempotent (bounded) integral residuated lattices are more commonly known as *Brouwerian algebras* (*Heyting algebras*).

A residuated lattice is *fully distributive* if its lattice reduct is distributive and moreover it satisfies the equations

$$x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z), \qquad (x \wedge y) \cdot z = (x \cdot z) \wedge (y \cdot z).$$

The variety of fully distributive IRLs (ICRLs) will be denoted by FdIRL (FdICRL).

A residuated lattice is *semilinear* if it is isomorphic to a subdirect product of totally ordered residuated lattices. Semilinear residuated lattices form a variety whose finitely subdirectly irreducible algebras are precisely the totally orderfed residuated lattices. The subvariety of semilinear CRLs will be denoted by SemCRL. Abelian ℓ-groups and Abelian ℓ-group cones are subvarieties of SemCRL, while MV-algebras are a variety of bounded semilinear CRLs.

Lemma 2.5 ([40, Proposition 2.3])**.** *A* CRL $\mathbf{A}$ *is finitely subdirectly irreducible if and only if* 1 *is join irreducible in* $\mathbf{A}$*:* $x \vee y = 1$ *in* $\mathbf{A}$ *implies that either* $x = 1$ *or* $y = 1$.

Theorem 2.6 (EDPC in varieties of CRLs [25, Theorem 3.55])**.**
A variety $\mathsf{V} \subseteq \mathsf{CRL}$ *has EDPC if and only if there is some* $n \in \mathbb{N}$ *such that* V *satisfies the equation* $(1 \wedge x)^{n+1} = (1 \wedge x)^n$.

2.4. **Conuclear residuated lattices.** A *conucleus* on a residuated lattice $\mathbf{A}$ is an interior operator $\Box$ on $\mathbf{A}$ whose image is a submonoid of $\mathbf{A}$, or equivalently an interior operator $\Box$ which satisfies $\Box x \cdot \Box y \leq \Box(x \cdot y)$ and $\Box(1) = 1$. A *conuclear residuated lattice* is an expansion $\langle \mathbf{A}, \Box \rangle$ of a residuated lattice $\mathbf{A}$ by a conucleus $\Box$. More precisely, we call $\langle \mathbf{A}, \Box \rangle$ a *conuclear expansion* of $\mathbf{A}$. Given a class K of residuated lattices, a *conuclear* K*-algebras* is a conuclear expansion of a K-algebra, i.e. a conuclear residuated lattice of the form $\langle \mathbf{A}, \Box \rangle$ for $\mathbf{A} \in \mathsf{K}$.

The image of $\Box$, or equivalently the set of all fixpoints $a = \Box(a)$ of $\Box$, carries the structure of a residuated lattice $\mathbf{A}_\Box$. This is a sub-sℓ-monoid of $\mathbf{A}$ but its meet $\wedge_\Box$ and its division operations $\backslash_\Box$ and $/_\Box$ need not coincide with those of $\mathbf{A}$:

$$x \wedge_\Box y := \Box(x \wedge y), \qquad x \backslash_\Box y := \Box(x \backslash y), \qquad x /_\Box y := \Box(x/y).$$

Residuated lattices of the form $\mathbf{A}_\Box$ are called the *conuclear images* of $\mathbf{A}$.

Given a class K of residuated lattices, $\mathbb{C}(\mathsf{K})$ denotes the class of all conuclear images of K-algebras and $\mathbb{C}\mathrm{x}(\mathsf{K})$ denotes the class of all conuclear expansions of K-algebras. Because the identity map is a conucleus, $\mathsf{K} \subseteq \mathbb{C}(\mathsf{K})$ and each residuated lattice is the reduct of some conuclear residuated lattice.

The map $x \mapsto 1 \wedge x$ is a conucleus on every residuated lattice $\mathbf{A}$. Its conuclear image is called the *negative cone* of $\mathbf{A}$ and denoted by $\mathbf{A}^-$. Given a class of residuated lattices K, the class of negative cones of K-algebras will be denoted by K^-. A conucleus $\Box$ on $\mathbf{A}$ will be called *negative* if $\Box x \leq 1 \wedge x$, or equivalently if $\mathbf{A}_\Box \subseteq \mathbf{A}^-$. Given a class K of residuated lattices, the class of all expansions of algebras in K by a negative conucleus will be denoted by $\mathbb{C}\mathrm{x}^-(\mathsf{K})$.

Clearly if K is a universal class or a (quasi)variety of residuated lattices, then so is $\mathbb{C}\mathrm{x}(\mathsf{K})$ and $\mathbb{C}\mathrm{x}^-(\mathsf{K})$, since these classes are axiomatized by adding the equational definition of a (negative) conucleus to an axiomatization of K.

As a general pattern, we shall derive results about $\mathbb{C}(\mathsf{K})$ from results about $\mathbb{C}\mathrm{x}(\mathsf{K})$. To this end, it will be useful to make explicit the relevant translation.

Lemma 2.7 (Conuclear translation)**.**
Given a conuclear residuated lattice $\langle \mathbf{A}, \Box \rangle$, there is a translation τ_{cn} from terms in the signature of (bounded) residuated lattices to terms in the signature of (bounded) conuclear residuated lattices such that for each universal sentence ϕ

$$\mathbf{A}_\Box \vDash \phi \iff \langle \mathbf{A}, \Box \rangle \vDash \tau_{\mathrm{cn}}(\phi),$$

namely τ_{cn} commutes with equality and with logical connectives and

$$\tau_{\mathrm{cn}}(1) := 1, \qquad \tau_{\mathrm{cn}}(\bot) := \bot, \qquad \tau_{\mathrm{cn}}(\top) := \Box\top,$$

$$\begin{aligned} \tau_{\mathrm{cn}}(x) &:= \Box x \text{ for each variable } x, \\ \tau_{\mathrm{cn}}(t \circ y) &:= \tau_{\mathrm{cn}}(t) \circ \tau_{\mathrm{cn}}(u) \text{ for } \circ \in \{\vee, \cdot\}, \\ \tau_{\mathrm{cn}}(t \circ y) &:= \Box(\tau_{\mathrm{cn}}(t) \circ \tau_{\mathrm{cn}}(u)) \text{ for } \circ \in \{\wedge, \backslash, /\}. \end{aligned}$$

Proof. This is straightforward given the definition of $\mathbf{A}_\Box$. ☐

Lemma 2.8 (FEP for $\mathbb{C}(\mathsf{K})$)**.**
Let K be a class of residuated lattices. If $\mathbb{C}\mathrm{x}(\mathsf{K})$ has the FEP, then so does $\mathbb{C}(\mathsf{K})$.

Proof. This is an immediate consequence of Lemma 2.7. ☐

Lemma 2.9. *A conuclear* CRL *$\langle \mathbf{A}, \Box \rangle$ is finitely subdirectly irreducible if and only if $\Box x \vee \Box y = 1$ for $x, y \leq 1$ in $\langle \mathbf{A}, \Box \rangle$ implies that either $\Box x = 1$ or $\Box y = 1$, or equivalently if and only if 1 is join irreducible in $\mathbf{A}_\Box$.*

Proof. The proof is a straightforward modification of the proof of Lemma 2.5, so we merely sketch it. The lattice of congruences of each conuclear CRL is isomorphic to the lattice of its *multiplicative $\Box$-filters*: multiplicative filters closed under $\Box$. Let $\mathrm{Fg}^{\mathbf{A}}_* a$ denote the principal multiplicative filter generated by $a \in \mathbf{A}$ and let $\mathrm{Fg}^{\mathbf{A}}_\Box a$ denote the principal multiplicative $\Box$-filter generated by a. It is known that $\mathrm{Fg}^{\mathbf{A}}_* a = \mathrm{Fg}^{\mathbf{A}}_*(1 \wedge a)$ for all $a \in \mathbf{A}$ and $\mathrm{Fg}^{\mathbf{A}}_* a \cap \mathrm{Fg}^{\mathbf{A}}_* b = \mathrm{Fg}^{\mathbf{A}}_*(a \vee b)$ for $a, b \leq 1$ in $\mathbf{A}$. Moreover, $\mathrm{Fg}^{\mathbf{A}}_\Box = \mathrm{Fg}^{\mathbf{A}}_* \Box(1 \wedge a)$. Thus $\mathrm{Fg}^{\mathbf{A}}_\Box 1 = \mathrm{Fg}^{\mathbf{A}}_\Box a \cap \mathrm{Fg}^{\mathbf{A}}_\Box b$ if and only if $1 = \Box(1 \wedge a) \vee \Box(1 \wedge b)$. Consequently, $\mathbf{A}$ is finitely subdirectly irreducible if and only if $\Box(1 \wedge a) \vee \Box(1 \wedge b) = 1$ implies $\Box(1 \wedge a) = 1$ or $\Box(1 \wedge b) = 1$, in other words if and only if $\Box a \vee \Box b = 1$ implies $\Box a = 1$ or $\Box b = 1$. ☐

Theorem 2.10 (EDPC in varieties of conuclear CRLs [45])**.**
A variety $\mathsf{V} \subseteq \mathbb{C}\mathsf{x}(\mathsf{CRL})$ *has EDPC if and only if there is some* $n \in \mathbb{N}$ *such that* V *satisfies the equation* $(\Box(1 \wedge x))^{n+1} = (\Box(1 \wedge x))^n$.

2.5. **Words and multisets.** A *word* over a set X is a finite sequence of elements of X, written as $[x_1, \dots, x_n]$. Words over X form a pomonoid $\operatorname{Word} X$ with concatenation of words as the monoidal operation, the empty word $[]$ as the monoidal unit, and the subword relation as the partial order: $u \sqsubseteq v$ if and only if u may be obtained by removing some (not necessarily contiguous) letters from v.

A *multiset* over a set X is a function $f\colon X \to \mathbb{N}$ with finite support, i.e. with only finitely many $x \in X$ such that $f(x) \neq 0$. Multisets over X form a commutative pomonoid $\operatorname{Multi} X$ with componentwise addition $\oplus$ as the monoidal operation, the empty multiset $[]$ as the monoidal unit, and the submultiset relation as the partial order: $f \sqsubseteq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$.

We define the notation $[x_1, \dots, x_n, y]$ inductively, with $[]$ being the base case: $[x_1, \dots, x_n, y](y) := [x_1, \dots, x_n](y) + 1$ and $[x_1, \dots, x_n, y](z) := [x_1, \dots, x_n](z)$ for each $z \neq y$. Given $n \in \mathbb{N}$, we use the notation $nw := w \oplus \dots \oplus w$ (n times).

Each map of sets $f\colon X \to Y$ induces a map $\operatorname{Multi} f\colon \operatorname{Multi} X \to \operatorname{Multi} Y$, which is an order preserving homomorphism:

$$(\operatorname{Multi} f)([x_1, \dots, x_n]) := [f(x_1), \dots, f(x_n)].$$

In practice, we shall write simply $f([x_1, \dots, x_n])$ instead of $(\operatorname{Multi} f)([x_1, \dots, x_n])$.

If X is a subset of an integral commutative pomonoid $\mathbf{M}$, then we define the map $-^{\mathbf{M}}\colon \operatorname{Multi} X \to \mathbf{M}$ as

$$[x_1, \dots, x_n]^{\mathbf{M}} := x_1 \cdot^{\mathbf{M}} \dots \cdot^{\mathbf{M}} x_n.$$

This map is well-defined because $\mathbf{M}$ is commutative, and it is order inverting because $\mathbf{M}$ is integral. That is,

$$u \sqsubseteq v \text{ in } \operatorname{Multi} X \implies v^{\mathbf{M}} \leq^{\mathbf{M}} u^{\mathbf{M}}.$$

2.6. **Higman's Lemma.** A key technical tool repeatedly employed in this paper will be Higman's Lemma [28]. For the sake of simplicity, we only state it for algebras in a finite signature.[3]

A poset P satisfies *ascending chain condition* if it has no infinite ascending chains, i.e. there is no infinite sequence $x_0 < x_1 < \dots$ with $x_0, x_1, \dots \in X$. It is *dually well partially ordered* if each downset of P is finitely generated, or equivalently if P has no infinite antichains and P satisfies the ascending chain condition.

Lemma 2.11 (Higman's Lemma [28])**.**
Let $\mathbf{A}$ *be an algebra in a finite algebraic signature and let* $\leq$ *be a partial order on* $\mathbf{A}$ *such that for each function symbol* f *of arity* n*:*

(i) *if* $a_1 \leq b_1, \dots, a_n \leq b_n$*, then* $f(a_1, \dots, a_n) \leq f(b_1, \dots, b_n)$*,*
(ii) $f(a_1, \dots, a_n) \leq a_i$ *for each* $i \in \{1, \dots, n\}$*.*

[3]While Higman's original result was stated in terms well partially ordered sets, in this paper it will be convenient to state its order dual version. Higman's original lemma was also formulated in terms of 0-generated algebras, i.e. algebras generated by their primitive constants. However, in practice it is more convenient to talk about, for example, monoids generated by a (dually) well partially ordered set rather than about 0-generated algebras in a signature which expands the signature of monoids by some set of constants.

If $\mathbf{A}$ *is generated by a set* $X \subseteq \mathbf{A}$ *which is dually well partially ordered by* $\leq$, *then* $\mathbf{A}$ *itself is dually well partially ordered by* $\leq$.

In the case of monoids, Higman's Lemma is precisely the claim that each integral pomonoid generated by a dually well partially ordered subset is itself dually well partially ordered (compare [23, Lemma 4.2]). In particular, each finitely generated integral pomonoid is dually well partially ordered.

Finitely generated integral sℓ-monoids will play an important role in this paper. These are covered by the following lemma, since each sℓ-monoid generated by a subset X is generated as a join semilattice by the submonoid generated by X.

Lemma 2.12 ([23, Lemma 4.3])**.**
Let $\mathbf{A}$ *be a join semilattice generated by a dually well partially ordered subset. Then* $\mathbf{A}$ *satisfies the ascending chain condition. In particular, each finitely generated integral s*ℓ*-monoid satisfies the ascending chain condition.*

3. Conuclear expansions and π-embeddings

Our first step will be to gain a better understanding of the class operator $\mathbb{UCxS}$. Rather like $\mathbf{A} \in \mathbb{U}(\mathsf{K})$ holds if each finite partial subalgebra has an embedding into K (Theorem 2.2), we show that $\mathbb{CxS}(\mathbf{A}) \subseteq \mathbb{UCxS}(\mathsf{K})$ holds if each finite partial subalgebra has a well-behaved embedding into $\mathsf{K} \subseteq \mathsf{IRL}$ (Lemma 3.7), namely what we call a π-embedding. We then use this to show (Theorem 3.9) that $\mathbb{CxU}(\mathsf{K}) = \mathbb{UCx}(\mathsf{K})$, provided that $\mathsf{K} \subseteq \mathsf{IRL}$ is closed under subalgebras and $\mathbb{U}(\mathsf{K})$ has locally finite monoid reducts. Consequently (Theorem 3.10), if a universal class K with locally finite monoid reducts has the FEP, then so do $\mathbb{Cx}(\mathsf{K})$ and $\mathbb{C}(\mathsf{K})$. This for example implies that the variety of conuclear n-potent integral commutative residuated lattices has the FEP (Corollary 3.11). A similar result (Theorem 3.6) for monadic integral residuated lattices (defined below) states that if $\mathsf{K} \subseteq \mathsf{IRL}$ is a locally finite universal class, then the universal class of all monadic expansions of algebras in K has the FEP. This for example implies that the variety of monadic Gödel algebras has the FEP (which was already proved in [14]).

Let us repeat here the remark from the introduction that the results proved here apply equally well to *bounded* integral residuated lattices, with virtually identical proofs. Indeed, the bounded case is very slightly simpler, since we can do away with the discussion of coinitiality (each *bounded* sub-sℓ-monoid is a coinitial subset). We omit explicit discussion of the bounded case altogether, relying on the interested reader to observe that at no point would the presence of the bottom constant incur any additional complications.

We start by observing (Lemma 3.2) that within the set $\mathbb{CxS}(\mathbf{A})$ we may restrict to finite partial subalgebras of a particular kind. Given an integral residuated lattice $\mathbf{A}$ and a finite set $S \subseteq \mathbf{A}$, let

$$\langle S \rangle^{\mathbf{A}} := \text{sub-s}\ell\text{-monoid of } \mathbf{A} \text{ generated by } S.$$

That is, $\langle S \rangle^{\mathbf{A}}$ consists of non-empty finite joins of finite products of elements of S. We show that $\langle S \rangle^{\mathbf{A}}$ is (the sℓ-monoid reduct of) a residuated lattice in $\mathbb{CS}(\mathbf{A})$.

To prevent a potential misreading, we emphasize that when we talk about a coinitial set $S \subseteq X$ with $X \subseteq \mathbf{A}$ in the following, we always mean a set S which is coinitial as a subset of X (rather than a conitial subset of $\mathbf{A}$).

Lemma 3.1 (The conuclear IRL $\langle \mathrm{Sg}^{\mathbf{A}}\, X, \Box_S \rangle$)**.**
Consider $\mathbf{A} \in \mathsf{IRL}$*, a set* $X \subseteq \mathbf{A}$*, and a finite coinitial* $S \subseteq X$*. Then* $\langle S \rangle^{\mathbf{A}}$ *is the image of a conucleus* $\Box_S$ *on* $\mathrm{Sg}^{\mathbf{A}}\, X$*.*

Proof. We first show that $\langle S \rangle^{\mathbf{A}}$ is coinitial in $\mathrm{Sg}^{\mathbf{A}}\, X$. Each $a \in \mathrm{Sg}^{\mathbf{A}}\, X$ has the form $a = t^{\mathbf{A}}(x_1, \dots, x_n)$ for some residuated lattice term t and $x_1, \dots, x_n \in X$. Induction over the complexity of t shows that some element of $\langle S \rangle^{\mathbf{A}}$ lies below a. The base case is precisely the assumption that S is coinitial in X. The case of the constant 1 is trivial. Finally, in the inductive step for $\circ \in \{\wedge, \vee, \cdot, \backslash, /\}$ we merely observe that if $a_1 \leq t_1^{\mathbf{A}}(x_1, \dots, x_n)$ and $a_2 \leq t_2^{\mathbf{A}}(x_1, \dots, x_n)$ for some $a_1, a_2 \in \langle S \rangle^{\mathbf{A}}$, then $a_1 \cdot a_2 \leq t_1^{\mathbf{A}}(x_1, \dots, x_n) \circ^{\mathbf{A}} t_2^{\mathbf{A}}(x_1, \dots, x_n)$. Notice that the cases $\circ \in \{\wedge, \backslash, /\}$ rely on the integrality of $\mathbf{A}$.

Because $\langle S \rangle^{\mathbf{A}}$ satisfies the ascending chain condition by Lemma 2.12, each element of ${\downarrow} a \cap \langle S \rangle^{\mathbf{A}}$ for $a \in \mathrm{Sg}^{\mathbf{A}}\, X$ lies below a maximal element of ${\downarrow} a \cap \langle S \rangle^{\mathbf{A}}$. Moreover, ${\downarrow} a \cap \langle S \rangle^{\mathbf{A}}$ is non-empty because $\langle S \rangle^{\mathbf{A}}$ is coinitial in $\mathrm{Sg}^{\mathbf{A}}\, X$, and it is upward directed because $\langle S \rangle^{\mathbf{A}}$ is a sub-sℓ-monoid of $\mathrm{Sg}^{\mathbf{A}}\, X$. Thus ${\downarrow} a \cap \langle S \rangle^{\mathbf{A}}$ has exactly one maximal element. That is, $\langle S \rangle^{\mathbf{A}}$ is the image of an interior operator on $\mathrm{Sg}^{\mathbf{A}}\, X$, which will be denoted by $\Box_S$. Because $\langle S \rangle^{\mathbf{A}}$ is a submonoid of $\mathrm{Sg}^{\mathbf{A}}\, X$, the interior operator $\Box_S$ is a conucleus. □

The domain of the conucleus $\Box_S$ depends on X, so strictly speaking one ought to write $\Box_S^X$. We choose to suppress the superscript and simply write $\Box_S$. This is harmless because the conuclei $\Box_S^X$ agree with each other in the sense that if S is a coinitial subset of both X_1 and X_2, then $\Box_S^{X_1}(a) = \Box_S^{X_2}(a)$ for $a \in \mathrm{Sg}^{\mathbf{A}}\, X_1 \cap \mathrm{Sg}^{\mathbf{A}}\, X_2$.

Lemma 3.2 (Finite partial subalgebras of conuclear IRLs)**.**
Consider a conucleus $\Box$ *on* $\mathbf{A} \in \mathsf{IRL}$*. Then each finite partial subalgebra of* $\langle \mathbf{A}, \Box \rangle$ *is a restriction of* $\langle \mathrm{Sg}^{\mathbf{A}}\, X, \Box_S \rangle|_X$ *for some finite* $X \subseteq \mathbf{A}$ *and coinitial* $S \subseteq X$*.*

Proof. Consider a finite partial subalgebra $\langle \mathbf{A}, \Box \rangle|_Y$ of $\langle \mathbf{A}, \Box \rangle$. Take $X := Y \cup \Box[Y]$ and $S := X \cap \mathbf{A}_{\Box}$. Clearly $\mathbf{A}|_Y$ is a restriction of $\mathrm{Sg}^{\mathbf{A}}\, X|_X$ and S is coinitial in X. The map $\Box_S$ is thus a well-defined conucleus on $\mathrm{Sg}^{\mathbf{A}}\, X$ by Lemma 3.1. To show that $\langle \mathbf{A}, \Box \rangle|_Y$ is a restriction $\langle \mathrm{Sg}^{\mathbf{A}}\, X, \Box_S \rangle|_X$, it will suffice to prove that $\Box_S(a) = \Box(a)$ for each $a \in Y$. Because $\mathbf{A}_{\Box}$ is a sub-sℓ-monoid of $\mathbf{A}$ and $S \subseteq \mathbf{A}_{\Box}$, we have $\langle S \rangle^{\mathbf{A}} \subseteq \mathbf{A}_{\Box}$, so $\Box_S(a) \leq \Box(a)$. Conversely, $\Box(a) \leq a$ and $\Box(a) \in S$, so $\Box(a) = \Box_S(\Box(a)) \leq \Box_S(a)$. □

In particular, a universal sentence ϕ fails in $\langle \mathbf{A}, \Box \rangle$ if and only if it fails in some partial algebra $\langle \mathrm{Sg}^{\mathbf{A}}\, X, \Box_S \rangle|_X$ for some finite $X \subseteq \mathbf{A}$ and coinitial $S \subseteq X$, in the sense that there is a valuation of variables v in X with respect to which each subterm of ϕ is well-defined in $\langle \mathrm{Sg}^{\mathbf{A}}\, X, \Box_S \rangle|_X$ and ϕ comes out false.

Theorem 3.3 (FEP for conuclear locally finite K-algebras)**.**
Let $\mathsf{K} \subseteq \mathsf{IRL}$ *be a locally finite universal class. Then* $\mathbb{C}\mathsf{x}(\mathsf{K})$ *and* $\mathbb{C}(\mathsf{K})$ *have the FEP.*

Proof. The claim for $\mathbb{C}\mathsf{x}(\mathsf{K})$ is an immediate consequence of Lemma 3.2. The claim for $\mathbb{C}(\mathsf{K})$ then follows from Lemma 2.8. □

The above theorem covers, among other examples, the varieties of conuclear Boolean algebras (i.e. S4 modal algebras), conuclear Gödel algebras, and conuclear n-valued MV-algebras. On the other hand, it does not cover the case of the variety of conuclear bounded ICRLs, which was shown to have the FEP by Amano [2] (see also [44, Theorem 13]) using the methodology of Blok and van Alten [11].

An analogous theorem holds for the class mK introduced in [18], which arises in the study of one-variable fragments of first-order substructural logics. Algebras in the class mK will be called *weakly monadic* K-algebras here. While in some cases (such as for Heyting algebras) these coincide with monadic K-algebras as normally understood, in other cases they form a strictly larger class. For example, monadic Gödel algebras are precisely the weakly monadic Gödel algebras which satisfy the constant domain axiom $\forall(\forall x \vee y) = \forall x \vee \forall y$ (cf. [18, Example 2.3]).

More precisely, by a *weakly monadic pair* on a residuated lattice $\mathbf{A}$ we mean a conucleus $\forall$ and a closure operator $\exists$ such that $\mathbf{A}_\forall$ is a subalgebra of $\mathbf{A}$ and moreover $\forall$ and $\exists$ have the same image: $\exists[\mathbf{A}] = \forall[\mathbf{A}]$. This last condition is equivalent to $\forall$ and $\exists$ being *adjoint*:

$$a \leq \forall b \iff \exists a \leq b \qquad \text{for all } a, b \in \mathbf{A}.$$

A *monadic residuated lattice* is then a residuated lattice equipped with a weakly monadic pair. The class of weakly monadic residuated lattices forms a variety.

Lemma 3.4 (The weakly monadic IRL $\langle \mathrm{Sg}^{\mathbf{A}} X, \forall_S, \exists_S \rangle$)**.**
Consider a locally finite $\mathbf{A} \in$ IRL, a set $X \subseteq \mathbf{A}$, and a finite coinitial $S \subseteq X$. Then $\mathrm{Sg}^{\mathbf{A}} S$ is the image of a weakly monadic pair $\forall_S$ and $\exists_S$ on $\mathrm{Sg}^{\mathbf{A}} X$.

Proof. $\mathrm{Sg}^{\mathbf{A}} S$ is coinitial in $\mathrm{Sg}^{\mathbf{A}} X$ because $\langle S \rangle^{\mathbf{A}} \subseteq \mathrm{Sg}^{\mathbf{A}} S$ and $\langle S \rangle^{\mathbf{A}}$ is coinitial in $\mathrm{Sg}^{\mathbf{A}} X$ by Lemma 3.1. The set $\mathrm{Sg}^{\mathbf{A}} S$ is finite because $\mathbf{A}$ is locally finite. Because $\mathrm{Sg}^{\mathbf{A}} S$ is finite and closed under finite meets, it is the image of a closure operator $\exists_S$ on $\mathrm{Sg}^{\mathbf{A}} X$. Because $\mathrm{Sg}^{\mathbf{A}} S$ is finite, coinitial, and closed under binary joins, it is the image of an interior operator $\forall_S$ on $\mathrm{Sg}^{\mathbf{A}} X$. Moreover, $\forall_S$ is a conucleus since $\mathrm{Sg}^{\mathbf{A}} S$ is a submonoid of $\mathrm{Sg}^{\mathbf{A}} X$. Because $\forall_S$ and $\exists_S$ have the same image and this image is a subalgebra of $\mathrm{Sg}^{\mathbf{A}} X$, they form a weakly monadic pair. ☐

Lemma 3.5 (Finite partial subalgebras of locally finite weakly monadic IRLs)**.**
Consider a weakly monadic pair $\forall$ and $\exists$ on a locally finite $\mathbf{A} \in$ IRL. Then each finite partial subalgebra of $\langle \mathbf{A}, \forall, \exists \rangle$ is a restriction of the finite algebra $\langle \mathrm{Sg}^{\mathbf{A}} X, \forall_S, \exists_S \rangle$ for some finite $X \subseteq \mathbf{A}$ and coinitial $S \subseteq X$.

Proof. Consider a finite partial subalgebra $\langle \mathbf{A}, \forall, \exists \rangle|_Y$ of $\langle \mathbf{A}, \forall, \exists \rangle$. Take $X := Y \cup \forall[Y] \cup \exists[Y]$ and $S := X \cap \forall[\mathbf{A}] = X \cap \exists[\mathbf{A}]$. Clearly $\mathbf{A}|_Y$ is a restriction of $\mathrm{Sg}^{\mathbf{A}} X|_X$ and S is coinitial in X. Then $\mathrm{Sg}^{\mathbf{A}} S$ is the image of a weakly monadic pair $\forall_S$ and $\exists_S$ on $\mathrm{Sg}^{\mathbf{A}} X$ by Lemma 3.4. To show that $\langle \mathbf{A}, \forall, \exists \rangle|_Y$ is a restriction $\langle \mathrm{Sg}^{\mathbf{A}} X, \forall_S, \exists_S \rangle|_X$, it will suffice to prove that $\forall_S(a) = \forall(a)$ and $\exists_S(a) = \exists(a)$ for each $a \in X$. Because $\forall[\mathbf{A}] = \exists[\mathbf{A}]$ is a subalgebra of $\mathbf{A}$ and $S \subseteq \forall[\mathbf{A}] = \exists[\mathbf{A}]$, we have $\mathrm{Sg}^{\mathbf{A}} S \subseteq \forall[\mathbf{A}] = \exists[\mathbf{A}]$, so $\forall_S(a) \leq \forall(a)$ and $\exists(a) \leq \exists_S(a)$. Conversely, $\forall(a) \leq a$ and $\forall(a) \in S$, so $\forall(a) = \forall_S(\forall(a)) \leq \forall_S(a)$. Similarly, $a \leq \exists(a)$ and $\exists(a) \in S$, so $\exists(a) = \exists_S(\exists(a)) \geq \exists_S(a)$. ☐

Theorem 3.6 (FEP for weakly monadic locally finite K-algebras)**.**
Let $\mathsf{K} \subseteq$ IRL be a locally finite universal class. Then weakly monadic K-algebras have the FEP.

Proof. This is an immediate consequence of Lemma 3.5. ☐

The above theorem generalizes the corresponding result for varieties of Heyting algebras proved by Bezhanishvili [7, Theorem 44].[4] For example, weakly monadic Gödel algebras have the FEP, since Gödel algebras are locally finite.[5]

We now relax the requirement that K-algebras be locally finite. Instead, we shall require that they have locally finite monoid reducts, or equivalently that they have locally finite sℓ-monoid reducts. We show that if such a universal class $\mathsf{K} \subseteq \mathsf{IRL}$ has the FEP, then $\mathbb{C}\mathrm{x}(\mathsf{K})$ and $\mathbb{C}(\mathsf{K})$ inherit this property.

Consider an integral residuated lattice $\mathbf{A}$ and a finite set $X \subseteq \mathbf{A}$. An embedding $\iota\colon \mathbf{A}|_X \hookrightarrow \mathbf{B}$ into an integral residuated lattice $\mathbf{B}$ will be called a *π-embedding* if for all words $w \in \mathrm{Word}\, X$ and all $a \in X$

$$w^{\mathbf{A}} \leq^{\mathbf{A}} a \implies \iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a).$$

Recall that we use the notation $w^{\mathbf{A}} := x_1 \cdot \ldots \cdot x_n$ for $w := [x_1, \ldots, x_n]$. More explicitly, the above condition states that for all $x_1, \ldots, x_n \in X$ and all $a \in X$

$$x_1 \cdot^{\mathbf{A}} \ldots \cdot^{\mathbf{A}} x_n \leq^{\mathbf{A}} a \iff \iota(x_1) \cdot \ldots \cdot \iota(x_n) \leq^{\mathbf{B}} \iota(a).$$

Lemma 3.7 ($\mathbb{UC}\mathrm{x}\mathbb{S}$ and π-embeddings)**.**
Suppose that each finite partial subalgebra $\mathbf{A} \in \mathsf{IRL}$ has a π-embedding into a class $\mathsf{K} \subseteq \mathsf{IRL}$. Then $\mathbb{C}\mathrm{x}\mathbb{S}(\mathbf{A}) \subseteq \mathbb{UC}\mathrm{x}\mathbb{S}(\mathsf{K})$.

Proof. By Theorem 2.2 we need to prove that each finite partial subalgebra of each algebra in $\mathbb{C}\mathrm{x}\mathbb{S}(\mathbf{A})$ embeds into $\mathbb{C}\mathrm{x}\mathbb{S}(\mathsf{K})$. By Lemma 3.2 each such finite partial subalgebra is a restriction of $\langle \mathrm{Sg}^{\mathbf{A}} X, \Box_S \rangle|_X$ for some finite $X \subseteq \mathbf{A}$ and coinitial $S \subseteq X$. It therefore suffices to embed each $\langle \mathrm{Sg}^{\mathbf{A}} X, \Box_S \rangle|_X$ into $\mathbb{C}\mathrm{x}\mathbb{S}(\mathsf{K})$.

By assumption there is a π-embedding ι of $\mathbf{A}|_X$ into some $\mathbf{B} \in \mathsf{K}$, which is therefore a π-embedding of $(\mathrm{Sg}^{\mathbf{A}} X)|_X$ into $\mathbf{B}$. It suffices to show that ι is an embedding of the partial algebra $\langle \mathrm{Sg}^{\mathbf{A}} X, \Box_S \rangle|_X$ into $\langle \mathrm{Sg}^{\mathbf{B}} \iota[X], \Box_{\iota[S]} \rangle \in \mathbb{C}\mathrm{x}\mathbb{S}(\mathsf{K})$.

Suppose therefore that $\Box_S(a) = b$ for $a, b \in X$. We show that $\Box_{\iota[S]}(\iota(a)) = \iota(b)$. The equality $\Box_S(a) = b$ implies that $b \leq^{\mathbf{A}} a$ and $b \in S$, so $\iota(b) \leq^{\mathbf{B}} \iota(a)$ and $\iota(b) \in \iota[S]$. It remains to show that $y \leq^{\mathbf{B}} \iota(a)$ implies $y \leq^{\mathbf{B}} \iota(b)$ for each $y \in \langle \iota[S] \rangle^{\mathbf{B}}$. Equivalently, it remains to show that $\iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a)$ implies $\iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(b)$ for all $w \in \mathrm{Word}\, S$. But this holds because ι is a π-embedding: $\iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a)$ implies that $w^{\mathbf{A}} \leq^{\mathbf{A}} b$, which implies that $w^{\mathbf{A}} \leq^{\mathbf{A}} b$ because $\Box_S(a) = b$ and $w \in \mathrm{Word}\, S$, so $\iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(b)$. ☐

Remark 3.8. It is not difficult to formulate a *necessary* and sufficient condition for $\mathbb{C}\mathrm{x}\mathbb{S}(\mathbf{A}) \subseteq \mathbb{UC}\mathrm{x}\mathbb{S}(\mathsf{K})$ in the same spirit, namely that for all $a \leq^{\mathbf{A}} b$ in X with $a \in S$ if $w^{\mathbf{A}} \leq^{\mathbf{A}} b$ implies $w^{\mathbf{A}} \leq^{\mathbf{A}} a$ for all $w \in \mathrm{Word}\, S$, then $\iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(b)$ implies $\iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a)$ for all $w \in \mathrm{Word}\, S$. Inspecting the proof of Lemma 3.7 shows that it was in fact this condition which we used to establish the inclusion $\mathbb{C}\mathrm{x}\mathbb{S}(\mathbf{A}) \subseteq \mathbb{UC}\mathrm{x}\mathbb{S}(\mathsf{K})$. We omit the straightforward proof that this condition is also necessary, since we have no further use for this necessary and sufficient condition.

[4] Theorem 44 of [7] talks about the Finite Model Property rather than the Finite Embeddability Property, but these properties are equivalent for varieties of monadic Heyting algebras by Theorem 2.4, since monadic Heyting algebras have EDPC by the correspondence between congruences and monadic filters established in [7].

[5] A variety V in a finite signature is locally finite if and only if for each $n \in \mathbb{N}$ there is a finite bound on the cardinality of n-generated subdirectly irreducible algebras in V [8]. But subdirectly irreducible Gödel algebras are chains and n-generated Gödel chains have cardinality at most $n + 2$.

Theorem 3.9 ($\mathbb{UC}\mathrm{x} = \mathbb{C}\mathrm{x}\mathbb{U}$ in the case of locally finite monoid reducts)**.**
Consider a class $\mathsf{K} \subseteq \mathsf{IRL}$ *which is closed under subalgebras. If* $\mathbb{U}(\mathsf{K})$ *has locally finite monoid reducts, then*

$$\mathbb{C}\mathrm{x}\mathbb{U}(\mathsf{K}) = \mathbb{UC}\mathrm{x}(\mathsf{K}).$$

Proof. The class $\mathbb{C}\mathrm{x}\mathbb{U}(\mathsf{K})$ is a universal class, since it is axiomatized by adding the inequalities defining a conucleus to any axiomatization of $\mathbb{U}(\mathsf{K})$. Thus $\mathbb{UC}\mathrm{x}(\mathsf{K}) \subseteq \mathbb{UC}\mathrm{x}\mathbb{U}(\mathsf{K}) = \mathbb{C}\mathrm{x}\mathbb{U}(\mathsf{K})$. Conversely, we need to prove that $\mathbb{C}\mathrm{x}\mathbb{U}(\mathsf{K}) \subseteq \mathbb{UC}\mathrm{x}(\mathsf{K})$. Since $\mathbb{U}(\mathsf{K}) = \mathbb{SU}(\mathsf{K})$, by Lemma 3.7 it suffices to show that for each finite partial subalgebra $\mathbf{A}|_X$ of each $\mathbf{A} \in \mathbb{U}(\mathsf{K})$ has a π-embedding into some $\mathbf{B} \in \mathsf{K}$. Because $\mathbf{A}$ has a locally finite monoid reduct, $\langle X \rangle^{\mathbf{A}}$ is finite. Taking $Y := \langle X \rangle^{\mathbf{A}}$, there is an embedding ι of $\mathbf{A}|_Y$ into $\mathbf{B} \in \mathsf{K}$ by Theorem 2.2. This embedding restricts to a π-embedding of $\mathbf{A}|_X$ into $\mathbf{B}$ because $w^{\mathbf{A}} \in Y$ for each $w \in \mathrm{Word}\, X$. □

Theorem 3.10 (FEP for conuclear IRLs with locally finite monoid reducts)**.**
Consider a universal class $\mathsf{K} \subseteq \mathsf{IRL}$ *with locally finite monoid reducts. If* K *has the FEP, then so do* $\mathbb{C}\mathrm{x}(\mathsf{K})$ *and* $\mathbb{C}(\mathsf{K})$*.*

Proof. Applying Theorem 3.9 to the class of finite algebras in K yields the claim for $\mathbb{C}\mathrm{x}(\mathsf{K})$. The claim for $\mathbb{C}(\mathsf{K})$ then follows by Lemma 2.8. □

Each universal class of IRLs with locally finite monoid reducts is n-potent for some $n \in \mathbb{N}$ (it satisfies $x^{n+1} = x^n$). Conversely, each n-potent commutative monoid is locally finite. More generally, each n-potent monoid which satisfies one of the generalized commutativity equations of [15] is locally finite [1].

Corollary 3.11 (FEP for n-potent conuclear ICRLs)**.**
The variety of conuclear n-potent ICRL*s has the FEP for each* $n \in \mathbb{N}$*.*

Proof. The variety of n-potent ICRLs is known to have the FEP [11, Theorem 4.2]. The result now follows from Theorem 3.10. □

The case $n := 1$ for bounded ICRLs states that conuclear Heyting algebras have the FEP, which was recently proved by de Groot and Shillito [21, Corollary 4.11].[6]

It remains to consider how the conucleus $\Box_S$ is actually computed. In the remainder of this section, it will be convenient to treat $\Box_S a$ as a partial operation on $\mathbf{A}$, which is defined if and only if $\langle S \rangle^{\mathbf{A}} \cap {\downarrow} a$ is non-empty.

Lemma 3.12. *Consider* $\mathbf{A} \in \mathsf{ICRL}$ *with no infinite ascending chains and with computable primitive operations. Each decision procedure for the condition* $\omega a \leq b$ *for* $a, b \in \mathbf{A}$ *yields a computation procedure for the partial map* $\langle S, a \rangle \mapsto \Box_S a$*, where* $S := \{s_1, \ldots, s_n\} \subseteq \mathbf{A}$*.*

Proof. The value $\Box_S a$ is well-defined if and only if $\langle S \rangle^{\mathbf{A}} \cap {\downarrow} a$ is non-empty, which in turn holds if and only if the decidable condition $\omega s \leq a$ holds, where s is some product containing each element of S at least once.

If $\Box_S a$ exists, then it is the join of all products in the monoid generated by S which lie below a. For each $s \in S$ we may divide these products into those

[6]More precisely, they proved that the logic of conuclear Heyting algebras enjoys the FMP with respect to a certain relational semantics. But their relational frames give rise to conuclear Heyting algebras and the class of conuclear Heyting algebras is the equivalent algebraic semantics of the logic of conuclear Heyting algebras, so their result implies the algebraic FMP for the variety of conuclear Heyting algebras. The FEP then follows from Theorem 2.4 because conuclear Heyting algebras have EDPC by Theorem 2.10.

products $s \cdot p$ which contain s as a factor and those products which do not. Then $s \cdot p \leq a$ if and only if $p \leq s\backslash a$, so the join of all the products of the form $s \cdot p$ is in fact $s \cdot \Box_S(s\backslash a)$. This yields the following recursive procedure for computing $\Box_S a$:

$$\Box_S a = \begin{cases} s \cdot \Box_S(s\backslash a) \vee \Box_{S-\{s\}} a & \text{if } \Box_{S-\{s\}} a \text{ exists,} \\ s \cdot \Box_S(s\backslash a) & \text{otherwise.} \end{cases}$$

If the left-hand side exists, then so does the right-hand side. Because $a \leq s\backslash a$ and $\mathbf{A}$ has no infinite ascending chains and S is finite, we may iterate this reduction until we obtain a join where the well-defined operation $\Box_T x$ occurs only in contexts where either $T = \emptyset$ or $t\backslash x = x$ for each $t \in T$. In both cases $x = 1$ and $\Box_T x = 1$: in the former case this is trivial, and in the latter case $t_1 \cdot \ldots \cdot t_i \leq x$ for $t_1, \ldots, t_i \in T$ implies that $1 \leq t_i\backslash(\ldots(t_1\backslash x)) = x$. This terminates the computation of $\Box_S a$. □

The above lemma provides a simple recursive procedure for computing $\Box_S$. However, it also has two limitations. Firstly, while it applies to the algebras $(\mathbb{Z}^-)^k$ and therefore suffices for the purposes of studying conuclei on Abelian ℓ-group cones (Section 5), it does not apply to $\mathrm{Lex}(n, \mathbb{Z})$ and therefore it does not suffice for the purposes of studying $\vee$-conuclei on Abelian ℓ-group cones (Section 8). Secondly, we do not see a simple way to adapt it to handle the meet preserving conucleus $\Box_S^{\wedge}$ (Section 7). Accordingly, we also provide a more general, brute force procedure for computing $\Box_S$, which avoids these limitations.

Lemma 3.13. *Consider $\mathbf{A} \in \mathsf{IRL}$ with computable primitive operations. Each pair of decision procedures for the conditions $\langle s_1, \ldots, s_n \rangle^{\mathbf{A}} \cap {\downarrow} a = \emptyset$ for $a, s_1, \ldots, s_n \in \mathbf{A}$ and $\langle s_1, \ldots, s_n \rangle^{\mathbf{A}} \cap ({\downarrow} a - {\downarrow} b) = \emptyset$ for $b \leq a$ in $\mathbf{A}$ and $s_1, \ldots, s_n \in \mathbf{A}$ yields a computation procedure for the partial map $\langle S, a \rangle \mapsto \Box_S a$, where $S := \{s_1, \ldots, s_n\} \subseteq \mathbf{A}$.*

Proof. If $\langle s_1, \ldots, s_n \rangle^{\mathbf{A}} \cap {\downarrow} a = \emptyset$, then $\Box_S a$ is undefined. Otherwise, take $i := 0$ and enumerate elements of $\langle S \rangle^{\mathbf{A}}$ until some $b_0 \in \langle S \rangle^{\mathbf{A}}$ with $b_0 \leq a$ is reached. Then repeat the following while $\langle S \rangle^{\mathbf{A}}$ intersects ${\downarrow} a - {\downarrow} b_i$: enumerate elements of the submonoid of $\mathbf{A}$ generated by S until finding some $c \in {\downarrow} a - {\downarrow} b_i$. Then take $b_{i+1} := b_i \vee c$ and increment i. Clearly $b_{i+1} \in \langle S \rangle^{\mathbf{A}}$. Because $\langle S \rangle^{\mathbf{A}}$ satisfies the ascending chain condition by Lemma 2.12, this process eventually terminates: otherwise $b_0 < b_1 < \ldots$ would be an infinite ascending chain. The final b_i is then the value of $\Box_S a$, since $b_i \in \langle S \rangle^{\mathbf{A}}$ and $\langle S \rangle^{\mathbf{A}}$ does not intersect ${\downarrow} a - {\downarrow} b_i$, i.e. each element of $\langle S \rangle^{\mathbf{A}}$ below a in fact lies below b_i. □

4. Conuclear expansions and ω-embeddings

While the sufficient condition for the inclusion $\mathbb{C}\mathsf{xS}(\mathbf{A}) \subseteq \mathbb{UC}\mathsf{xS}(\mathsf{K})$ given in Lemma 3.7 was useful in the locally finite case, we will need to do some further work before we can apply it to the case of conuclear Abelian ℓ-group cones.

We show that the π-embeddings introduced in the previous section may be replaced by what we call ω-embeddings (Lemma 4.3), or equivalently by what we call weak ω-embeddings (Lemma 4.4). Given a variety $\mathsf{V} \subseteq \mathsf{ICRL}$, we then reduce the problem of finding an ω-embedding of each finite partial subalgebra of each $\mathbf{A} \in \mathsf{V}$ into some class of integral commutative residuated lattices to the case where $\mathbf{A}$ is finitely subdirectly irreducible (Theorem 4.11).

The reduction from π-embeddings to ω-embeddings will rely on Higman's Lemma for multisets, which is a particular instance of Lemma 2.11.

Lemma 4.1 (Higman's Lemma for multisets)**.**
Let X be a finite set. Then each upset of Multi X *is finitely generated.*

In contrast, downsets of Multi X need not be finitely generated. In fact, the finitely generated downsets of Multi X are precisely the finite downsets of Multi X. However, we show that downsets of Multi X are still finite unions of downsets of a special form. A downset D of Multi X will be called *quasi-principal* if there are $u, v \in$ Multi X such that for each $w \in$ Multi X

$$w \in D \iff w \sqsubseteq u \oplus mv \text{ for some } m \in \mathbb{N}.$$

It will be called *quasi-finitely generated* if it is a finite (possibly empty) union of quasi-principal downsets, or more explicitly if there are $u_1, v_1, \dots, u_k, v_k \in$ Multi X (possibly $k = 0$) such that

$$w \in D \iff w \sqsubseteq u_1 \oplus mv_1 \text{ or } \dots \text{ or } w \sqsubseteq u_k \oplus mv_k \text{ for some } m \in \mathbb{N}.$$

Each principal (finitely generated) set is quasi-principal (quasi-finitely generated).

Lemma 4.2 (Downsets of Multi X)**.**
Let X be a finite set. Then each downset of Multi X *is quasi-finitely generated.*

Proof. Consider a downset D of Multi X. If $D =$ Multi X, the claim holds for $k := 1$ if we take $u_1 := []$ and we take v_1 to be the multiset containing exactly one occurrence of each element of X. We may therefore assume that $D \subsetneq$ Multi X.

Consider the non-empty upset $U :=$ Multi $X - D$ of Multi X. By Higman's Lemma (Lemma 4.1) there is $l \geq 1$ and there are $w_1, \dots, w_l \in$ Multi X such that $w \in U$ if and only if $w_i \sqsubseteq w$ for some $i \in \{1, \dots, l\}$. Each of the conditions $w_i \sqsubseteq w$ is equivalent to the finite conjunction of inequalities $w_i(x) \leq w(x)$ for $x \in X$, so the condition $w \in U$ is equivalent to a non-empty finite disjunction of non-empty finite conjunctions of conditions of the form $p \leq w(x)$ for $x \in X$ and $p \in \mathbb{N}$.

Negating this finite disjunction and transforming the negation into disjunctive normal form yields that the condition $w \notin U$ is equivalent to a non-empty finite disjunction of non-empty finite conjunctions of conditions of the form $w(x) < p$ for $x \in X$ and $p > 1$, or equivalently to a non-empty finite disjunction of non-empty finite conjunctions of inequalities of the form $w(x) \leq p$ for $x \in X$ and $p \in \mathbb{N}$. We may further assume, removing redundant conditions if necessary, that in each such conjunction each $x \in X$ occurs in at most one inequality.

It now only remains to observe that each such finite conjunction of inequalities $w(x_i) \leq p_i$ for $x_1, \dots, x_n \in X$ and $p_1, \dots, p_n \in \mathbb{N}$ is equivalent to the condition that $w \sqsubseteq u \oplus mv$ for some $m \in \mathbb{N}$, where $u := p_1[x_1] \oplus \dots \oplus p_n[x_n]$, while v is the multiset containing exactly one occurrence of each element of $X - \{x_1, \dots, x_n\}$ and nothing else. Consequently, there are $u_1, v_1, \dots, u_k, v_k \in$ Multi X such that

$$w \notin U \iff w \sqsubseteq u_1 \oplus mv_1 \text{ or } \dots \text{ or } w \sqsubseteq u_k \oplus mv_k \text{ for some } m \in \mathbb{N}. \qquad \square$$

Given $\mathbf{A} \in \mathsf{ICRL}$ and $a, b, c \in \mathbf{A}$, it will be convenient to use the notation

$$a \cdot b^\omega \nleq^{\mathbf{A}} c \iff a \cdot b^m \nleq^{\mathbf{A}} c \text{ for all } m \in \mathbb{N},$$
$$a \cdot b^\omega \leq^{\mathbf{A}} c \iff a \cdot b^m \leq^{\mathbf{A}} c \text{ for some } m \in \mathbb{N}.$$

Consider $\mathbf{A}, \mathbf{B} \in \mathsf{ICRL}$ and a finite set $X \subseteq \mathbf{A}$. An embedding $\iota \colon \mathbf{A}|_X \hookrightarrow \mathbf{B}$ will be called an *ω-embedding* if

$$a \cdot b^\omega \nleq^{\mathbf{A}} c \implies \iota(a) \cdot \iota(b)^\omega \nleq^{\mathbf{B}} \iota(c) \qquad \text{for all } a, b, c \in \mathbf{A}.$$

It will be called a *weak ω-embedding* if

$$b^{\omega} \nleq^{\mathbf{A}} c \implies \iota(b)^{\omega} \nleq^{\mathbf{B}} \iota(c) \qquad \text{for all } b, c \in \mathbf{A}.$$

A well-defined composition of (weak) ω-embeddings is a (weak) ω-embedding.

Lemma 4.3 (From π-embeddings to ω-embeddings)**.**
Each finite partial subalgebra of $\mathbf{A} \in \mathsf{ICRL}$ has a π-embedding into $\mathsf{K} \subseteq \mathsf{ICRL}$ if and only if each finite partial subalgebra of $\mathbf{A}$ has an ω-embedding into K.

Proof. Left to right, each π-embedding is an ω-embedding.

Right to left, we extend each finite $X \subseteq \mathbf{A}$ to some finite $Y \subseteq \mathbf{A}$ such that each ω-embedding of $\mathbf{A}|_Y$ into an algebra $\mathbf{B} \in \mathsf{K}$ restricts to a π-embedding of $\mathbf{A}|_X$ into $\mathbf{B}$. To this end, it will suffice to find for each $a \in X$ a finite extension $Y_a \subseteq \mathbf{A}$ of X such that for each $w \in \operatorname{Multi} X$ and each ω-embedding ι of $\mathbf{A}|_{Y_a}$ into $\mathbf{B}$

$$(*) \qquad w^{\mathbf{A}} \leq^{\mathbf{A}} a \iff \iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a).$$

Taking $Y := \bigcup_{a \in X} Y_a$ then yields the desired extension of X.

Given $a \in X$, take

$$U_a := \{w \in \operatorname{Multi} X \mid w^{\mathbf{A}} \leq^{\mathbf{A}} a\}.$$

Because $\mathbf{A}$ is integral, U_a is an upset of $\operatorname{Multi} X$. Higman's Lemma (Lemma 4.1) applied to U_a now provides $w_1, \dots, w_l \in \operatorname{Multi} X$ such that for all $w \in \operatorname{Multi} X$

$$w \in U_a \iff w_1 \sqsubseteq w \text{ or } \dots \text{ or } w_l \sqsubseteq w.$$

On the other hand, Lemma 4.2 applied to the downset $\operatorname{Multi} X - U_a$ provides $u_1, v_1, \dots, u_k, v_k \in \operatorname{Multi} X$ such that for all $w \in \operatorname{Multi} X$

$$w \notin U_a \iff w \sqsubseteq u_1 \oplus m v_1 \text{ or } \dots \text{ or } w \sqsubseteq u_k \oplus m v_k \text{ for some } m \in \mathbb{N}.$$

In particular, $w_i^{\mathbf{A}} \leq^{\mathbf{A}} a$ for each $i \in \{1, \dots, l\}$, while $u_i^{\mathbf{A}} \cdot (v_i^{\mathbf{A}})^m \nleq^{\mathbf{A}} a$ for each $i \in \{1, \dots, k\}$ and $m \in \mathbb{N}$. We take

$$Y_a := X \cup \{u_1^{\mathbf{A}}, v_1^{\mathbf{A}}, \dots, u_k^{\mathbf{A}}, v_k^{\mathbf{A}}, w_1^{\mathbf{A}}, \dots, w_l^{\mathbf{A}}\}.$$

We now prove the left-to-right implication in $(*)$. Suppose that $w^{\mathbf{A}} \leq^{\mathbf{A}} a$ for $w \in \operatorname{Multi} X$. Then $w \in U_a$, so $w_i \sqsubseteq w$ for some $i \in \{1, \dots, l\}$, hence $\iota(w_i) \sqsubseteq \iota(w)$ and by integrality $\iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(w_i)^{\mathbf{B}}$. Because ι is an ω-embedding, the inequality $w_i^{\mathbf{A}} \leq^{\mathbf{A}} a$ implies that $\iota(w_i)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a)$, so $\iota(w)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(w_i)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a)$ as desired.

It remains to prove the right-to-left implication in $(*)$. Suppose that $w^{\mathbf{A}} \nleq^{\mathbf{A}} a$. Then $w \notin U$, so $w \sqsubseteq u_i \oplus m v_i$ for some $i \in \{1, \dots, k\}$ and $m \in \mathbb{N}$, hence $\iota(w) \sqsubseteq \iota(u_i) \oplus m\iota(v_i)$ and $\iota(u_i)^{\mathbf{B}} \cdot \iota(v_i^{\mathbf{B}})^m \leq^{\mathbf{B}} \iota(w)^{\mathbf{B}}$. Because ι is an ω-embedding, the inequalities $u_i^{\mathbf{A}} \cdot (v_i^{\mathbf{A}})^m \nleq^{\mathbf{A}} a$ for all $m \in \mathbb{N}$ imply that $\iota(u_i)^{\mathbf{B}} \cdot (\iota(v_i)^{\mathbf{B}})^m \nleq^{\mathbf{B}} \iota(a)$ for all $m \in \mathbb{N}$. Consequently, the inequality $\iota(u_i)^{\mathbf{B}} \cdot \iota(v_i^{\mathbf{B}})^m \leq^{\mathbf{B}} \iota(w)^{\mathbf{B}}$ implies that $\iota(w)^{\mathbf{B}} \nleq^{\mathbf{B}} \iota(a)$ as desired. □

We can replace ω-embeddings by weak ω-embeddings in the above lemma, as we now show. Indeed, when dealing with the particular case of Abelian ℓ-group cones in the next section, we shall exclusively work with weak ω-embeddings. However, there are two reasons to work with ω-embeddings in this section. Firstly, the reduction from ω-embeddings to weak ω-embeddings relies on the presence of residuals, but we wish to write our proofs in such a way that the reader may immediately apply them to structures where residuals need not exist, such as expansions of sℓ-monoids by a conucleus, should they wish to do so. Secondly, we also wish to make the task of extending these proofs to meet-preserving conuclear expansions in Section 7 as straightforward as possible.

Lemma 4.4. *Consider* $\mathbf{A} \in \mathsf{ICRL}$ *and* $\mathsf{K} \subseteq \mathsf{ICRL}$. *Each finite partial subalgebra of* $\mathbf{A}$ *has an* ω*-embedding into* K *if and only if each finite partial subalgebra* $\mathbf{A}|_X$ *of* $\mathbf{A}$ *has a weak* ω*-embedding into* K.

Proof. Left to right, apply the hypothesis to $\mathbf{A}|_{X\cup\{1\}}$ and take $a := 1$. Right to left, consider a finite partial subalgebra $\mathbf{A}|_X$ and take $Y := X \cup \{a\backslash b \mid a, b \in X\}$. By assumption, $\mathbf{A}|_Y$ has an embedding ι into some $\mathbf{B} \in \mathsf{K}$ such that $b^\omega \not\leq^{\mathbf{A}} c$ implies $\iota(b)^\omega \not\leq^{\mathbf{B}} \iota(c)$ for $b, c \in Y$. We show that ι restricts to an ω-embedding of $\mathbf{A}|_X$ into $\mathbf{B}$. For $a, b, c \in X$ clearly $a \cdot b^\omega \not\leq^{\mathbf{A}} c$ if and only if $b^\omega \not\leq^{\mathbf{A}} a\backslash c$ and likewise $\iota(a) \cdot \iota(b)^\omega \not\leq^{\mathbf{B}} \iota(c)$ if and only if $\iota(b)^\omega \not\leq^{\mathbf{B}} \iota(a)\backslash\iota(c) = \iota(a\backslash c)$, using the fact that $a\backslash c \in Y$. But by assumption $b^\omega \not\leq^{\mathbf{A}} a\backslash c$ if and only if $\iota(b)^\omega \not\leq^{\mathbf{B}} \iota(a\backslash c)$. ☐

Theorem 4.5 ($\mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}$ and ω-embeddings)**.**
Suppose that each finite partial subalgebra of $\mathbf{A} \in \mathsf{ICRL}$ *has a (weak)* ω*-embedding into a class* $\mathsf{K} \subseteq \mathsf{ICRL}$. *Then* $\mathbb{C}\mathrm{x}\mathbb{S}(\mathbf{A}) \subseteq \mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathsf{K})$.

Proof. This is an immediate consequence of Lemmas 3.7, 4.3, and 4.4. ☐

In the remainder of this section, we show that for the purposes of finding a generating set of $\mathbb{C}\mathrm{x}(\mathsf{V})$ for a variety $\mathsf{V} \subseteq \mathsf{ICRL}$, it will suffice to consider ω-embeddings of $\mathsf{V}_{\mathrm{fsi}}$-algebras.

Lemma 4.6. *Let* $(\mathbf{A}_i)_{i\in I}$ *be a finite family of integral commutative residuated lattices, let* $\mathbf{A} := \prod_{i\in I} \mathbf{A}_i$*, and let* $\pi_i\colon \mathbf{A} \to \mathbf{A}_i$ *be the projection maps. Then*

$$a \cdot b^\omega \leq^{\mathbf{A}} c \iff \pi_i(a) \cdot \pi_i(b)^\omega \leq^{\mathbf{A}_i} \pi_i(c) \text{ for each } i \in I.$$

Proof. If $a \cdot b^n \leq^{\mathbf{A}} c$, then clearly $\pi_i(a) \cdot \pi_i(b)^n \leq^{\mathbf{A}_i} \pi_i(c)$ for each $i \in I$. Conversely, suppose that for each $i \in I$ there is some $n_i \in \mathbb{N}$ such that $\pi_i(a) \cdot \pi_i(b)^{n_i} \leq^{\mathbf{A}_i} \pi_i(c)$. Taking $n := \max\{n_i \mid i \in I\}$, which exists because I is finite, $\pi_i(a) \cdot \pi_i(b)^n \leq^{\mathbf{A}_i} \pi_i(c)$ for each $i \in I$, so $a \cdot b^n \leq^{\mathbf{A}} c$. ☐

Lemma 4.7 ([13, Lemma IV.6.6])**.** *Let* $\mathcal{F}$ *be a family of subsets of* I *such that*

(i) $I \in \mathcal{F}$,
(ii) if $J \subseteq K \subseteq I$ *and* $J \in \mathcal{F}$, *then* $K \in \mathcal{F}$,
(iii) if $J \cup K \in \mathcal{F}$, *then either* $J \in \mathcal{F}$ *or* $K \in \mathcal{F}$.

Then there is an ultrafilter $\mathcal{U}$ *on* I *such that* $\mathcal{U} \subseteq \mathcal{F}$.

Lemma 4.8 (Reduction to finite subdirect products)**.**
$\mathbb{C}\mathrm{x}\mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{K}) \subseteq \mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}\mathbb{P}_{\mathrm{fin}}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$ *for each class* $\mathsf{K} \subseteq \mathsf{ICRL}$.

Proof. By Theorem 4.5 it suffices to show that each finite partial subalgebra $\mathbf{A}|_X$ of each product $\mathbf{A} := \prod_{i\in I} \mathbf{A}_i$ with $\mathbf{A}_i \in \mathsf{K}$ has an ω-embedding ι into a finite product $\mathbf{B}$ of algebras in $\mathbb{I}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$.

Given $J \subseteq I$, let $\pi_J\colon \prod_{i\in I} \mathbf{A}_i \to \prod_{j\in J} \mathbf{A}_j$ be the restriction map. Because X is finite, there is a finite set $J \subseteq I$ such that π_J is an embedding of $\mathbf{A}|_X$ into $\prod_{j\in J} \mathbf{A}_j$. We now find an algebra $\mathbf{C} \in \mathbb{P}_{\mathrm{fin}}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$ and a homomorphism $h\colon \mathbf{A} \to \mathbf{C}$ such that

$$a \cdot (b^{\mathbf{A}})^\omega \not\leq^{\mathbf{A}} c \implies h(u_i)^{\mathbf{C}} \cdot (h(b)^{\mathbf{C}})^\omega \not\leq^{\mathbf{C}} h(c) \qquad \text{for all } i \in \{1, \ldots, k\}.$$

The map $\iota \times h\colon \mathbf{A}|_X \hookrightarrow \mathbf{B}$ for $\mathbf{B} := \prod_{j \in J} \mathbf{A}_j \times \mathbf{C} \in \mathbb{P}_{\text{fin}}\mathbb{I}\mathbb{P}_{\text{U}}(\mathsf{K})$ will then satisfy the requirements of the previous paragraph.

For each $a \in X$ and each $i \in \{1, \dots, k\}$ such that $a \cdot (b^{\mathbf{A}})^{\omega} \not\leq^{\mathbf{A}} c$ we shall define an algebra $\mathbf{C}_{a,b,c} \in \mathbb{P}_{\text{U}}(\mathsf{K})$ and a homomorphism $h_{a,b,c}\colon \mathbf{A} \to \mathbf{C}_{a,b,c}$ such that $h_{a,b,c}(c) \cdot h_{a,b,c}(b^{\mathbf{A}})^{\omega} \not\leq^{\mathbf{C}_{a,b,c}} h_{a,b,c}(c)$. We take $\mathbf{C}$ to be the product of the algebras $\mathbf{C}_{a,b,c}$ and $h\colon \mathbf{A} \to \mathbf{C}$ to be the product of the homomorphisms $h_{a,b,c}$. Then $\mathbf{C}$ and h will satisfy the requirements of the previous paragraph.

Let $\mathcal{F}_{a,b,c}$ be the family of all $J \subseteq I$ such that $\pi_J(c) \cdot \pi_J(b^{\mathbf{A}})^{\omega} \not\leq^{\pi_J[\mathbf{A}]} \pi_J(c)$. We define $\mathbf{C}_{a,b,c}$ to be an ultrapower $\prod_{i \in I} \mathbf{A}_i / \mathcal{U}_{a,b,c}$ with respect to an ultrafilter $\mathcal{U}_{a,b,c} \subseteq \mathcal{F}_{a,b,c}$, with $h_{a,b,c}\colon \mathbf{A} \to \mathbf{C}_{a,b,c}$ being the quotient map.

We use Lemma 4.7 to show that such an ultrafilter exists. Because $a \cdot (b^{\mathbf{A}})^{\omega} \not\leq^{\mathbf{A}} c$, we have $I \in \mathcal{F}_{a,b,c}$. Clearly if $J \in \mathcal{F}_{a,b,c}$ and $J \subseteq K \subseteq I$, then $K \in \mathcal{F}_{a,b,c}$. Finally, if $J \cup K \in \mathcal{F}_{a,b,c}$, then either $J \in \mathcal{F}_{a,b,c}$ or $K \in \mathcal{F}_{a,b,c}$: contrapositively, if $J, K \notin \mathcal{F}_{a,b,c}$, then there are $m, n \in \mathbb{N}$ such that $\pi_J(c) \cdot \pi_J(b^{\mathbf{A}})^m \leq^{\pi_J[\mathbf{A}]} \pi_J(c)$ and $\pi_K(c) \cdot \pi_K(b^{\mathbf{A}})^n \leq^{\pi_K[\mathbf{A}]} \pi_K(c)$, so $\pi_{J \cup K}(c) \cdot \pi_{J \cup K}(b^{\mathbf{A}})^p \leq^{\pi_{J \cup K}[\mathbf{A}]} \pi_{J \cup K}(c)$ for $p := \max(m, n)$, hence $J \cup K \notin \mathcal{F}_{a,b,c}$.

It remains to show that $h_{a,b,c}(c) \cdot h_{a,b,c}(b^{\mathbf{A}})^{\omega} \not\leq^{\mathbf{C}_{a,b,c}} h_{a,b,c}(c)$. Suppose otherwise. Then there is be some $n \in \mathbb{N}$ such that $h_{a,b,c}(c) \cdot h_{a,b,c}(b^{\mathbf{A}})^n \leq^{\mathbf{C}_{a,b,c}} h_{a,b,c}(c)$, and therefore some $J \in \mathcal{U}_{a,b,c}$ such that $\pi_J(c) \cdot \pi_J(b^{\mathbf{A}})^n \leq^{\pi_J[\mathbf{A}]} \pi_J(c)$. But then $J \notin \mathcal{F}_{a,b,c}$, contradicting the inclusion $\mathcal{U}_{a,b,c} \subseteq \mathcal{F}_{a,b,c}$. □

Theorem 4.9 (Generating class for conuclear V-algebras)**.**
Consider a variety $\mathsf{V} \subseteq \mathsf{ICRL}$. The variety of conuclear V-algebras is generated as a universal class by the conuclear expansions of $\mathbb{ISP}_{\text{fin}}(\mathsf{V}_{\text{fsi}})$:

$$\mathbb{C}\text{x}(\mathsf{V}) = \mathbb{U}\mathbb{C}\text{x}\mathbb{S}\mathbb{P}_{\text{fin}}(\mathsf{V}_{\text{fsi}}).$$

Proof. Each variety V satisfies $\mathsf{V} = \mathbb{ISP}(\mathsf{V}_{\text{fsi}})$. Moreover, for $\mathsf{V} \subseteq \mathsf{ICRL}$ we have $\mathbb{P}_{\text{U}}(\mathsf{V}_{\text{fsi}}) = \mathbb{P}_{\text{U}}(\mathsf{V} \cap \mathsf{ICRL}_{\text{fsi}}) \subseteq \mathbb{P}_{\text{U}}(\mathsf{V}) \cap \mathbb{P}_{\text{U}}(\mathsf{ICRL}_{\text{fsi}}) \subseteq \mathsf{V} \cap \mathsf{ICRL}_{\text{fsi}} = \mathsf{V}_{\text{fsi}}$: the inclusion $\mathbb{P}_{\text{U}}(\mathsf{ICRL}_{\text{fsi}}) \subseteq \mathsf{ICRL}_{\text{fsi}}$ holds because being finitely subdirectly irreducible is definable by a first-order condition in integral commutative residuated lattices (Lemma 2.5). Applying Lemma 4.8 to V_{fsi} thus proves the claim. □

Theorem 4.9 applies to any variety V of integral commutative residuated lattices. If we have further understanding of the finitely subdirectly irreducible algebras in V, we now show that we may in fact restrict to a subclass of V_{fsi}. This is indeed what we shall do in the next section in the case of Abelian ℓ-group cones.

Lemma 4.10. *Suppose that each finite partial subalgebra of each $\mathbf{A} \in \mathsf{K} \subseteq \mathsf{ICRL}$ has an ω-embedding into $\mathsf{L} \subseteq \mathsf{ICRL}$. Then $\mathbb{C}\text{x}\mathbb{S}\mathbb{P}_{\text{fin}}(\mathsf{K}) \subseteq \mathbb{U}\mathbb{C}\text{x}\mathbb{S}\mathbb{P}_{\text{fin}}(\mathsf{L})$.*

Proof. By Theorem 4.5 it suffices to show that for each finite product $\mathbf{A} := \prod_{i \in I} \mathbf{A}_i$ of algebras $\mathbf{A}_i \in \mathsf{K}$ with $i \in I$ and each finite $X \subseteq \mathbf{A}$ there is an ω-embedding ι of $\mathbf{A}|_X$ into some $\mathbf{B} \in \mathbb{S}\mathbb{P}_{\text{fin}}(\mathsf{L})$. Because I is finite, we may without loss of generality take $X := \prod_{i \in I} X_i$ for some family of finite sets $X_i \subseteq \mathbf{A}_i$ with $i \in I$. By assumption, each $\mathbf{A}_i|_{X_i}$ has an ω-embedding ι_i into L. Their product ι is then an embedding of $\mathbf{A}|_X$ into $\mathbb{S}\mathbb{P}_{\text{fin}}(\mathsf{L})$. Moreover, ι is an ω-embedding by Lemma 4.6. □

Theorem 4.11 (Generating class for conuclear V-algebras)**.**
Consider a variety $\mathsf{V} \subseteq \mathsf{ICRL}$ such that each finite partial subalgebra of each $\mathbf{A} \in \mathsf{V}_{\text{fsi}}$ has

a (weak) ω-embedding into $\mathsf{K} \subseteq \mathsf{V}$. Then the variety of conuclear V-algebras is generated as a universal class by conuclear $\mathbb{SP}_{\mathrm{fin}}(\mathsf{K})$-algebras:

$$\mathbb{C}\mathrm{x}(\mathsf{V}) = \mathbb{UC}\mathrm{x}\mathbb{SP}_{\mathrm{fin}}(\mathsf{K}).$$

Proof. This follows immediately from Theorem 4.9 and Lemma 4.10. □

5. Conuclear expansions of Abelian ℓ-group cones

We now move from the universal algebraic level to the particular case of Abelian ℓ-group cones. The goal of this section is to apply the main result of the previous section (Theorem 4.11) to the particular case of Abelian ℓ-group cones and thereby obtain a useful generating set for conuclear Abelian ℓ-group cones (Theorem 5.9). In doing so, we will rely on the Hahn representation of totally ordered Abelian groups (Theorem 5.1) in terms of real-valued functions on chains. Applying the restriction to finite partial algebras of the form $\langle \mathrm{Sg}^{\mathbf{A}} X, \Box_S \rangle$ (Lemma 3.2) to our generating set for conuclear Abelian ℓ-group cones, we then deduce that conuclear Abelian ℓ-group cones have a decidable universal theory (Theorem 5.10).

This will yield a number of further decidability results as corollaries. The universal theory of Abelian ℓ-groups equipped with a negative conucleus is also decidable (Theorem 5.15). Given that, by the Montagna–Tsinakis representation (Theorem 5.16), the conuclear images of Abelian ℓ-group cones are known to be precisely the cancellative ICRLs, we may further deduce the decidability of the universal theory of cancellative ICRLs (Theorem 5.18).

An *Abelian lattice-ordered group*, or *Abelian ℓ-group* for short, is an algebra of the form $\langle G, \wedge, \vee, +, 0, - \rangle$ such that $\langle G, \wedge, \vee \rangle$ is a lattice, $\langle G, +, 0, - \rangle$ is an Abelian group, and $\langle G, \vee, +, 0 \rangle$ is an sℓ-monoid. Each Abelian ℓ-group is fully distributive, i.e. it is a distributive lattice and satisfies

$$x + (y \wedge z) = (x + y) \wedge (x + z), \qquad (x \wedge y) + z = (x + z) \wedge (y + z).$$

It will be convenient to use additive notation for Abelian ℓ-groups because we will represent them by means of real-valued functions.

Abelian ℓ-groups form a variety AbLG which is term-equivalent to the variety of commutative residuated lattices axiomatized by the equation $x \cdot (x \backslash 1) = 1$: in one direction we take ($x \cdot y := x + y$ and $1 := 0$ and)

$$x \backslash y := -x + y, \qquad x / y := x - y,$$

while in the other direction we take ($x + y := x \cdot y$ and $0 := 1$ and)

$$-x := x \backslash 1 = 1/x.$$

A *totally ordered Abelian group* is an Abelian ℓ-group whose lattice reduct is a chain. These are precisely the finitely subdirectly irreducible Abelian ℓ-groups. A key example is the additive group of reals with the usual ordering:

$$\mathbb{R} := \langle \mathbb{R}, \min, \max, +, 0, - \rangle.$$

The additive groups $\mathbb{Q}$ and $\mathbb{Z}$ of rationals and integers are subalgebras of $\mathbb{R}$.

An *Abelian ℓ-group cone* is the negative cone $\mathbf{G}^-$ of some Abelian ℓ-group $\mathbf{G}$. In accordance with the additive notation for Abelian ℓ-groups, we will also use additive notation for Abelian ℓ-group cones, writing $x \ominus y$ for x / y. It was shown in [5] that Abelian ℓ-group cones form a subvariety AbLG^- of ICRL axiomatized by *cancellativity*, i.e. the equation $(x + z) \ominus (y + z) = x \ominus y$, and *divisibility*, i.e. (in the context of integral residuated lattices) the equation $(x \ominus y) + y = x \wedge y$. The cancellativity equation is equivalent to the cancellativity of the monoid reduct:

$$x + z = y + z \implies x = y.$$

The totally ordered Abelian ℓ-group cones form precisely the class of finitely subdirectly irreducible Abelian ℓ-group cones, denoted by $\mathsf{AbLG}^{-}_{\mathrm{fsi}}$.

Throughout this section, Γ denotes a chain. Given a totally ordered Abelian group $\mathbf{G}$, the set $\mathbf{G}^{\Gamma}$ of all functions $f\colon \Gamma \to \mathbf{G}$ is an Abelian group with group operations computed componentwise. The *support* of a function $f \in \mathbf{G}^{\Gamma}$ is the set

$$\operatorname{supp} f := \{p \in \Gamma \mid f(p) \neq 0\}.$$

The *lexicographic power* of $\mathbf{G}$ with respect to Γ is the following subgroup of $\mathbf{G}^{\Gamma}$:

$$\operatorname{Lex}(\Gamma, \mathbf{G}) := \{f \in \mathbf{G}^{\Gamma} \mid \operatorname{supp} f \text{ satisfies the ascending chain condition}\}.$$

$\operatorname{Lex}(\Gamma, \mathbf{G})$ has the structure of a totally ordered Abelian group with

$$f \leq g \text{ in } \operatorname{Lex}(\Gamma, G) \iff f(i) \leq g(i) \text{ for each } i \in \max \operatorname{supp}(g - f).$$

This is a total order because $\max \operatorname{supp}(g - f) = \max \operatorname{supp}(f - g)$ for $f, g \in \operatorname{Lex}(\Gamma, \mathbf{G})$ is either empty (if $f = g$) or it is a singleton. We use $\operatorname{Lex}(n, \mathbf{G})$ as an abbreviation for $\operatorname{Lex}(\Gamma, \mathbf{G})$ where $\Gamma := \{1, \dots, n\}$ is the canonical n-element chain.

Theorem 5.1 (The Hahn representation [27, 4])**.**
Each totally ordered Abelian group embeds into $\operatorname{Lex}(\Gamma, \mathbb{R})$ *for some chain* Γ.

In this section we shall use the additive notation $\omega f \leq g$ instead of $f^{\omega} \leq g$.

Lemma 5.2 (The relation $\omega f \nleq g$ on $\operatorname{Lex}^{-}(\Gamma, \mathbb{R})$)**.**
Consider $f, g < 0$ *in* $\operatorname{Lex}(\Gamma, \mathbb{R})$ *for some chain* Γ. *Then*

$$\omega f \nleq g \iff p < q \text{ for } \max \operatorname{supp} f = \{p\} \text{ and } \max \operatorname{supp} g = \{q\}.$$

Proof. Either $p < q$ or $p = q$ or $p > q$. If $p < q$, then $g(q) < 0 = nf(q)$ for all $n \in \mathbb{N}$, so $g < nf$ and $nf \nleq g$ in $\operatorname{Lex}^{-}(\Gamma, \mathbb{R})$, i.e. $\omega f \nleq g$. If $p = q$, then $nf(p) \leq g(p)$ for some $n \in \mathbb{N}$ by the Archimedean property of $\mathbb{R}$, so $nf \leq g$ in $\operatorname{Lex}^{-}(\Gamma, \mathbb{R})$ and $\omega f \leq g$. If $q < p$, then $f(p) < 0 = g(p)$, so $f \leq g$ in $\operatorname{Lex}^{-}(\Gamma, \mathbb{R})$ and $\omega f \leq g$. □

Lemma 5.3 (From chains to finite chains)**.**
Each finite partial subalgebra of $\operatorname{Lex}^{-}(\Gamma, \mathbb{R})$ *has a weak* ω*-embedding into* $\operatorname{Lex}^{-}(\Delta, \mathbb{R})$ *for some finite* $\Delta \subseteq \Gamma$.

Proof. Consider a finite set $X \subseteq \operatorname{Lex}^{-}(\Gamma, \mathbb{R})$. We show that for large enough finite $\Delta \subseteq \Gamma$ the restriction map $\rho_{\Delta}\colon \operatorname{Lex}^{-}(\Gamma, \mathbb{R}) \to \operatorname{Lex}^{-}(\Delta, \mathbb{R})$ restricts to a weak ω-embedding of $\operatorname{Lex}^{-}(\Gamma, \mathbb{R})|_X$ into $\operatorname{Lex}^{-}(\Delta, \mathbb{R})$. Because X is finite, the restriction $\rho_{\Delta}|_X$ is injective for large enough Δ. The map ρ_{Δ} preserves the group operations, since these are computed componentwise in $\operatorname{Lex}^{-}(\Gamma, \mathbb{R})$ and $\operatorname{Lex}^{-}(\Delta, \mathbb{R})$. If the singleton set $\max \operatorname{supp}(g - f)$ is included in Δ for each of the finitely many pairs $f \neq g$ in X, then $\rho_{\Delta}|_X$ is an order embedding on X. Finally, if the singleton set $\max \operatorname{supp} f$ is included in Δ for each of the finitely many $f \neq 0$ in X, then by Lemma 5.2 the map $\rho_{\Delta}|_X$ satisfies the implication

$$\omega f \nleq g \text{ in } \operatorname{Lex}^{-}(\Gamma, \mathbb{R}) \implies \omega \rho_{\Delta}(f) \nleq \rho_{\Delta}(g) \text{ in } \operatorname{Lex}^{-}(\Delta, R).$$ □

Lemma 5.4 (Rational solutions are dense)**.**
The set of rational solutions of each finite system of linear equalities and strict linear inequalities with rational coefficients is a dense subset of its set of real solutions.

Proof. Let us first consider the case where the system contains no inequalities. Gaussian elimination shows that either the solution set is empty or there are vectors $\mathbf{v}_0, \dots, \mathbf{v}_k$ with rational coefficients such that the solution set consists of all vectors of the form $\mathbf{v}_0 + \lambda_1 \mathbf{v}_1 + \dots \lambda_k \mathbf{v}_k$ for $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. The set of rational solutions is then the set of all vectors of this form with $\lambda_1, \dots, \lambda_k \in \mathbb{Q}$, which is clearly a dense subset of the solution set.

If the system contains some inequalities and $\mathbf{x}$ is a real solution of this system, then by the previous paragraph there are rational vectors arbitrarily close to $\mathbf{x}$ which satisfy all the equalities in the system. Moreover, since $\mathbf{x}$ satisfies all the strict linear inequalities of the system, these inequalities will be satisfied in each rational vector close enough to $\mathbf{x}$. Consequently, there are rational solutions of the system of equalities and inequalities arbitrarily close to $\mathbf{x}$. ☐

Lemma 5.5 (From $\mathbb{R}$ to $\mathbb{Z}$)**.**
Each finite partial subalgebra of $\mathrm{Lex}^-(n, \mathbb{R})$ *has a weak ω-embedding into* $\mathrm{Lex}^-(n, \mathbb{Z})$*.*

Proof. It will suffice to show that each finite partial subalgebra of $\mathrm{Lex}^-(n, \mathbb{R})$ has a weak ω-embedding into $\mathrm{Lex}^-(n, \mathbb{Q})$: clearly each finite partial subalgebra of $\mathrm{Lex}^-(n, \mathbb{Q})$ has a weak ω-embedding into $\mathrm{Lex}^-(n, \mathbb{Z})$ given by multiplication by any natural number which is sufficiently large in the divisibility order.

We first show that we need not be concerned with preserving residuals, i.e. with preserving the operation $\ominus$. Given a partial subalgebra $\mathbf{G}^-|_X$ of an Abelian ℓ-group cone $\mathbf{G}^-$, an embedding ι of $\mathbf{G}^-|_X$ into an Abelian ℓ-group cone $\mathbf{H}^-$ preserves the operation $\ominus$ if X is closed under binary meets and the embedding preserves addition and binary meets. To see this, consider $a, b \in \mathbf{G}^-$. The element $a \ominus^{\mathbf{G}} b = (a \wedge^{\mathbf{G}} b) \ominus^{\mathbf{G}} b$ satisfies $(a \ominus^{\mathbf{G}} b) +^{\mathbf{G}} (a \wedge^{\mathbf{G}} b) = a$ by the divisibility of $\mathbf{G}$, so $\iota(a \ominus^{\mathbf{G}} b) +^{\mathbf{H}} (\iota(a) \wedge^{\mathbf{H}} \iota(b)) = \iota(a)$. But $(\iota(a) \ominus^{\mathbf{H}} \iota(b)) +^{\mathbf{H}} (\iota(a) \wedge^{\mathbf{H}} \iota(b)) = \iota(a)$ by the divisibility of $\mathbf{H}$, so $\iota(a \ominus^{\mathbf{G}} b) = \iota(a) \ominus^{\mathbf{H}} \iota(b)$ by the cancellativity of $\mathbf{H}$.

Because each finite partial subalgebra of $\mathrm{Lex}^-(n, \mathbb{R})$ is a restriction of a finite partial subalgebra of $\mathrm{Lex}^-(n, \mathbb{R})$ which is closed under binary meets, it therefore suffices to show that each finite partial subalgebra of $\mathrm{Lex}^-(n, \mathbb{R})$ with universe $\{a_1, \dots, a_m\}$ has a weak ω-embedding into $\mathrm{Lex}^-(n, \mathbb{Q})$ with respect to the lattice and monoid operations. We do so by associating a finite system of linear equalities and strict inequalities with integer coefficients to this partial subalgebra.

The variables of this system will be $x_{i,p}$ for $i \in \{1, \dots, m\}$ and $p \in \{1, \dots, n\}$. By the original interpretation of variables we mean the interpretation of $x_{i,p}$ as $a_{i,p} := a_i(p)$. Whenever $a_i + a_j = a_k$, we add the equalities $x_{i,p} + x_{j,p} = x_{k,p}$ for $p \in \{1, \dots, n\}$. We also add all of the following (in)equalities which are satisfied under the original interpretation for $i, j \in \{1, \dots, m\}$ and $p, q \in \{1, \dots, n\}$:

$$\begin{array}{lll} x_{i,p} = x_{j,q}, & x_{i,p} < x_{j,q}, & x_{i,p} > x_{j,q}, \\ x_{j,p} = x_{k,q}, & x_{j,p} < x_{k,q}, & x_{j,p} > x_{k,q}, \\ x_{k,p} = x_{i,q}, & x_{k,p} < x_{i,q}, & x_{k,p} > x_{i,q}, \\ x_{i,p} = 0, & x_{i,p} < 0, & x_{i,p} > 0. \end{array}$$

The original interpretation $x_{i,p} := a_{i,p}$ is a real solution of this system. By Lemma 5.4 there is a rational solution $x_{i,p} := b_{i,p} \in \mathbb{Q}$ of this system arbitrarily closed to the original interpretation. Each such rational solution determines a partial subalgebra of $\mathrm{Lex}^-(n, \mathbb{Q})$ which is isomorphic to the original subalgebra of $\mathrm{Lex}^-(n, \mathbb{R})$ via the function ι mapping a_i to b_i, where $b_i(p) := b_{i,p}$. The fact that this is an bijection which preserves existing sums and the constant 0 is immediate. The fact this it is an order embedding follows from the fact that the validity of an inequality $a_i < a_j$ in $\mathrm{Lex}^-(n, \mathbb{R})$ corresponds to some set of equalities and inequalities in the above system: if $a_i < a_j$, then either $a_i(n) < a_j(n)$ or $a_i(n) = a_j(n)$ and $a_i(n-1) < a_j(n-1)$ or $\dots$, so one of these options is recorded in the above system of equalities and inequalities, hence $a_i < a_j$ in $\mathrm{Lex}^-(n, \mathbb{R})$ implies that $b_i < b_j$ in $\mathrm{Lex}^-(n, \mathbb{Q})$. Similarly, Lemma 5.2 ensures that the equalities and inequalities in our system record whether the condition $\omega a_i \nleq a_j$ holds, so $\omega a_i \nleq a_j$ in $\mathrm{Lex}^-(n, \mathbb{R})$ implies that $\omega b_i \nleq b_j$ in $\mathrm{Lex}^-(n, \mathbb{Q})$. ☐

Lemma 5.6. *Each finite partial subalgebra of* $\mathrm{Lex}^-(n, \mathbb{Z})$ *for each* $n \in \mathbb{N}$ *has a weak* ω*-embedding into* $(\mathbb{Z}^-)^n$.

Proof. Consider for $k \in \mathbb{N}$ the finite set

$$X_k := \{f \in \mathrm{Lex}^-(n, \mathbb{Z}) \mid -k \leq f(i) \leq k \text{ for each } i < n\}.$$

Because each finite subset of $\mathrm{Lex}^-(n, \mathbb{Z})$ is contained in X_k for some $k \in \mathbb{N}$, it will suffice to prove that the finite partial subalgebra of $\mathrm{Lex}^-(n, \mathbb{Z})$ with universe X_k has a weak ω-embedding into $(\mathbb{Z}^-)^n$ for each $k \in \mathbb{N}$. As in the proof of Lemma 5.5, it suffices to find an order embedding of X_k into $(\mathbb{Z}^-)^n$ which preserves addition.

We define the map $\iota\colon \mathrm{Lex}(n, \mathbb{Z}) \to \mathbb{Z}^n$ as

$$\iota\colon f \mapsto f(1)a_1 + \cdots + f(n)a_n$$

for some suitable $a_1, \dots, a_n \in \mathbb{Z}^n$ to be determined later. This map clearly preserves addition. We first show that ι indeed restricts to a map $\iota\colon X_k \to (\mathbb{Z}^-)^n$. That is, $f(1)a_1 + \cdots + f(n)a_n \leq 0$ in $\mathbb{Z}^n$ for $f \in X_k$. This holds trivially for $f = 0$. Otherwise, $\max \operatorname{supp} f = \{i\}$ for some $i \in \{1, \dots, n\}$ and $f(i) \leq -1$, so it suffices to prove that $a_i \geq f(1)a_1 + \cdots + f(i-1)a_{i-1}$ in $\mathbb{Z}^n$. Because $f \in X_k$, it suffices to choose $a_i \geq k(a_1 + \cdots + a_{i-1})$, in particular $a_1 \geq 0$.

To prove that ι restricted to X_k is an order embedding, we need to prove that

$$f(1)a_1 + \cdots + f(n)a_n \leq g(1)a_1 + \cdots + g(n)a_n \iff f \leq g \text{ in } \mathrm{Lex}(n, \mathbb{Z})$$

for all $f, g \in X_k$. Taking $h := g - f$, it suffices to prove that

$$0 \leq h(1)a_1 + \cdots + h(n)a_n \text{ in } \mathbb{Z}^n \iff 0 \leq h(i) \text{ for each } i \in \max \operatorname{supp} h.$$

If $f = g$, this holds trivially. Otherwise, let i be the unique element of $\max \operatorname{supp} h$. The above equivalence then states that

$$0 \leq h(1)a_1 + \cdots + h(i)a_i \iff 0 \leq h(i),$$

That is, $h(i) \geq 1$ implies $0 \leq h(1)a_1 + \cdots + h(i)a_i$, and $h(i) \leq -1$ implies $0 \nleq h(1)a_1 + \cdots + h(i)a_i$. Equivalently, $-h(1)a_1 - \cdots - h(i-1)a_{i-1} \leq a_i$, and $a_i \nleq h(1)a_1 + \cdots + h(i-1)a_{i-1}$. Because $|h(j)| \leq 2k$ for $j \in \{1, \dots, i-1\}$, this is equivalent to $2k(a_1 + \cdots + a_{i-1}) \leq a_i$, and $a_i \nleq 2k(a_1 + \cdots + a_{i-1})$. It therefore suffices to choose a_i so that $a_i > 2k(a_1 + \cdots + a_{i-1})$, in particular $a_1 > 0$.

Finally, we need to prove that $\omega f \not\leq g$ in $\mathrm{Lex}^-(n,\mathbb{Z})$ for $f,g \in X_k$ implies $\omega\iota(f) \not\leq \iota(g)$ in $(\mathbb{Z}^-)^n$. If $f = 0$ or $g = 0$, this holds trivially. Otherwise, $\omega f \not\leq g$ in $\mathrm{Lex}^-(n,\mathbb{Z})$ implies that $p < q$ for $\max \operatorname{supp} f = \{p\}$ and $\max \operatorname{supp} g = \{q\}$ by Lemma 5.2. Then $\iota(f) = f(1)a_1 + \cdots + f(p)a_p$ and $\iota(g) = g(1)a_1 + \cdots + g(q)a_q$. To ensure that $\omega\iota(f) \not\leq \iota(g)$, it suffices to choose $a_1,\ldots,a_n \geq 0$ in $\mathbb{Z}^n$ so that $\omega(-a_1 - \cdots - a_p) \not\leq -a_q$ for $q > p$. To this end, it suffices to choose $\pi_j(a_i) = 0$ for $j > i$ and $\pi_i(a_i) > 0$, where $\pi_j \colon \mathbb{Z}^n \to \mathbb{Z}$ denotes the projection maps.

The three constraints obtained above are easy to satisfy: take $c_1 := 1$ and $c_{i+1} := 2k(c_1 + \cdots + c_i) + 1$ and $a_i := \langle c_i,\ldots,c_i,0,\ldots,0\rangle$ with i non-zero components. □

Lemma 5.7 (From chains to $\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-)$)**.**
Each finite partial subalgebra of each totally ordered Abelian ℓ-group cone has a weak ω-embedding into $\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-)$.

Proof. Hahn's representation (Theorem 5.1) and Lemmas 5.3 and 5.5 imply that each finite partial subalgebra of $\mathbf{G}^-$ has a weak ω-embedding into $\mathrm{Lex}^-(n,\mathbb{Z})$ for some $n \in \mathbb{N}$, given that a well-defined composition of weak ω-embeddings is again a weak ω-embedding. □

Lemma 5.8. $\mathbb{ISP}_{\mathrm{fin}}(\mathbb{Z}) = \mathbb{IP}_{\mathrm{fin}}(\mathbb{Z})$*. Consequently,* $\mathbb{ISP}_{\mathrm{fin}}(\mathbb{Z}^-) = \mathbb{IP}_{\mathrm{fin}}(\mathbb{Z}^-)$*.*

Proof. The first claim follows from [9, Chapter XIII, Theorem 10], which states that each ℓ-group $\mathbf{G}$ such that $\mathbf{G}^-$ has no infinite ascending chains is isomorphic to

$$\oplus_{i\in I}\mathbb{Z} := \{f \in \mathbb{Z}^I \mid f(i) = 0 \text{ for all but finitely many } i \in I\}$$

for some set I.[7] Each $(\mathbb{Z}^k)^-$, hence each $\mathbf{G}^-$ for $\mathbf{G} \leq \mathbb{Z}^k$, satisfies the ascending chain condition, so $\mathbf{G} \cong \oplus_{i\in I}\mathbb{Z}$ for some set I. Moreover, $\mathbb{Z}^k$ and therefore also $\mathbf{G}$ satisfies the following universal sentence: if $a_1 \vee \cdots \vee a_{k+1} = 0$, then there is some $j \in \{1,\ldots,k+1\}$ such that $\bigvee_{i\neq j} a_i = 0$. Consequently, $|I| \leq k$ and $\mathbf{G} \cong \mathbb{Z}^I$.

To prove the second claim, it suffices to prove that each subalgebra of $\mathbb{Z}^-$ is the restriction to the negative cone of a subalgebra of $\mathbb{Z}$. To this end, observe that for each Abelian ℓ-group term $t(x_1,\ldots,x_n)$ there is some Abelian ℓ-group cone term $u(x_1,\ldots,x_n)$ such that $0 \wedge t^{\mathbf{G}}(a_1,\ldots,a_n) = u^{\mathbf{G}^-}(a_1,\ldots,a_n)$ for each Abelian ℓ-group $\mathbf{G}$ and $a_1,\ldots,a_n \in \mathbf{G}^-$. This holds because t is equal to a distributive lattice combination of Abelian group terms, so $0 \wedge t$ is equal to a distributive lattice combination of terms of the form $0 \wedge (x_1 + \cdots + x_m - y_1 - \cdots - y_n)$, and for all $a_1,\ldots,a_m,b_1,\ldots,b_n \in \mathbf{G}^-$

$$0 \wedge (a_1 + \cdots + a_m - b_1 - \cdots - b_n) = (a_1 + \cdots + a_m) \ominus^{\mathbf{G}^-} (b_1 + \cdots + b_n). \quad \square$$

Theorem 5.9 (Generation for conuclear Abelian ℓ-group cones)**.**
The variety of conuclear Abelian ℓ-group cones is generated as follows:

$$\mathbb{C}\mathsf{x}(\mathsf{AbLG}^-) = \mathbb{UC}\mathsf{x}\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-).$$

Proof. This follows from Lemma 5.7 and Theorem 4.11, since the finitely subdirectly irreducible Abelian ℓ-group cones are the totally ordered ones. Lemma 5.8 is used to reduce $\mathbb{SP}_{\mathrm{fin}}(\mathbb{Z}^-)$ to $\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-)$. □

Theorem 5.10 (Decidability for conuclear Abelian ℓ-group cones)**.**
The universal theory of conuclear Abelian ℓ-group cones is decidable.

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[7] We are grateful to Filip Jankovec for pointing out this reference to us.

Proof. Because the universal theory of conuclear Abelian ℓ-group cones is finitely axiomatizable, it is recursively enumerable. We show that its complement (in the set of all universal sentences) is also recursively enumerable.

Consider a universal sentence $\forall z_1 \dots \forall z_n\, \phi$, where ϕ is a quantifier-free formula in the signature of Abelian ℓ-group cones. By Theorem 5.9, ϕ fails to hold in some Abelian ℓ-group cone with a conucleus if and only if it fails to hold in $\mathbb{C}\mathrm{x}\mathbb{S}((\mathbb{Z}^-)^k)$ for some $k \in \mathbb{N}$. By Lemma 3.2 this happens if and only if there is a finite set $X \subseteq \mathbf{A} := (\mathbb{Z}^-)^k$ and coinitial $S \subseteq X$ such that the universal sentence fails in the partial algebra $\langle \mathrm{Sg}^{\mathbf{A}}\, X, \Box_S \rangle|_X$. This means that there is a map v evaluating the variables of ϕ in X such that the universal sentence ϕ is evaluated as false in $\langle \mathrm{Sg}^{\mathbf{A}}\, X, \Box_S \rangle|_X$ with respect to v. All the operations of $(\mathrm{Sg}^{\mathbf{A}}\, X)|_X$ are simply the restrictions of the computable operations of $(\mathbb{Z}^-)^k$ to the given finite set X, which are computable from the parameters k and X.

To prove that the complement of the universal theory of Abelian ℓ-group cones with a conucleus is recursively enumerable, it therefore suffices to show that the operation $\Box_S$ is computable from the parameters k, X, and S. This follows from Lemma 3.12, using the fact that the condition $\omega a \leq b$ for $a, b \in (\mathbb{Z}^-)^k$ has a uniform decision procedure for all $k \in \mathbb{N}$, namely it is equivalent to: $\pi_i(b) < 0$ implies $\pi_i(a) < 0$ for each $i \in \{1, \dots, k\}$.

Alternatively, this also follows from Lemma 3.13, using Lemma 5.11. While Lemma 5.11 is not needed in the current section, given the simpler argument in the previous paragraph, it will be needed in Sections 7 and 8. ☐

Lemma 5.11. *For $\mathbf{A} := \mathrm{Lex}^-(n, \mathbb{Z})^k$ there are decision procedures (uniform in k and n) for the conditions $\langle S \rangle^{\mathbf{A}} \cap {\downarrow} a = \emptyset$ for $a \in \mathbf{A}$ and $\langle S \rangle^{\mathbf{A}} \cap ({\downarrow} a - {\downarrow} b)$ for $b \leq a$ in $\mathbf{A}$.*

Proof. These problems are equivalent to the problem of deciding whether, given a pair $b \leq a$ in $\mathbf{A}$ (or given $a \in \mathbf{A}$) and a finite set $S \subseteq \mathbf{A}$, the submonoid of $\mathbf{A}$ generated by S intersects ${\downarrow} a - {\downarrow} b$ (intersects ${\downarrow} a$). This submonoid consists of the elements $\alpha_1 x_1 + \dots + \alpha_n x_m$ for $x_1, \dots, x_m \in X$ and $\alpha_1, \dots, \alpha_m \in \mathbb{N}$. We thus wish to decide whether there are $\alpha_1, \dots, \alpha_m \in \mathbb{N}$ such that $\alpha_1 x_1 + \dots + \alpha_m x_m \leq a$ but $\alpha_1 x_1 + \dots + \alpha_m x_m \nleq b$ (or merely such that $\alpha_1 x_1 + \dots + \alpha_m x_m \leq a$). This is some finite Boolean combination of conditions of the forms

$$\alpha_1 x_{1,i} + \dots + \alpha_m x_{m,i} < k,$$
$$\alpha_1 x_{1,i} + \dots + \alpha_m x_{m,i} > k,$$

where $x_{j,i} := x_j(i)$ for $j \in \{1, \dots, m\}$ and $i \in \{0, \dots, n-1\}$. Each such Boolean combination is equivalent to a finite disjunction of finite conjunctions of conditions of these forms. It therefore suffices to show that it is decidable whether there are $\alpha_1, \dots, \alpha_n \in \mathbb{N}$ satisfying a finite conjunction of conditions of this type.

This is straightforward. Given $y_1, \dots, y_m \in \mathbb{N}$ and $k \in \mathbb{N}$, the solution set of a condition of the form $\alpha_1 y_1 + \dots + \alpha_m y_m < k$ is empty if $y_j \geq k$ for all $j \in \{1, \dots, m\}$, otherwise it is a downset of $\mathbb{N}^n_\infty$ generated by a non-empty finite set which can be computed from the condition. The solution set of a condition of the form $\alpha_1 y_1 + \dots + \alpha_m y_m > k$ is empty if $y_j = 0$ for all $j \in \{1, \dots, m\}$, otherwise it is an upset of $\mathbb{N}^n_\infty$ generated by a non-empty finite set which can be computed from the condition. Finally, given a finite set of finitely generated downsets of $\mathbb{N}^n_\infty$ and a finite set of finitely generated upsets of $\mathbb{N}^n_\infty$, the problem of computing whether their intersection is non-empty reduces this to the problem of computing whether intersection of a finite set of principal downsets of $\mathbb{N}^n_\infty$ and a finite set of principal upsets of $\mathbb{N}^n_\infty$ is non-empty. But $\downarrow p_1 \cap \cdots \cap \downarrow p_i = \downarrow(p_1 \wedge \cdots \wedge p_i)$ and $\uparrow q_1 \cap \cdots \cap \uparrow q_j = \uparrow(q_1 \vee \cdots \vee q_j)$ for $p_1, \ldots, p_i, q_1, \ldots, q_j \in \mathbb{N}^n_\infty$, and $\downarrow p \cap \uparrow q$ is non-empty if and only if $q \leq p$. $\square$

It is now straightforward to derive an analogous generation and decidability result for Abelian ℓ-groups equipped with a negative conucleus.

The following lemma, if we ignore all mention of conuclei, was in effect established in [5, Section 7], using a somewhat different argument.

Lemma 5.12. *There is a computable translation τ_- from universal sentences in the signature of Abelian ℓ-group with a negative conucleus to the signature of Abelian ℓ-group cones with a conucleus such that $\langle \mathbf{G}, \square \rangle \vDash \phi$ if and only if $\langle \mathbf{G}^-, \square \rangle \vDash \tau_-(\phi)$.*

Proof. For each n-ary term t in the former signature there are by induction terms t_+ and t_- of arity $2n$ in the latter signature such that for each Abelian ℓ-group with a negative conucleus $\mathbf{G}$ and all $a_1, \ldots, a_n \in \mathbf{G}$

$$(0 \wedge t)^{\mathbf{G}}(a_1, \ldots, a_n) = t_+^{\mathbf{G}^-}(0 \wedge a_1, 0 \wedge -a_1, \ldots, 0 \wedge a_n, 0 \wedge -a_n),$$
$$(0 \wedge -t)^{\mathbf{G}}(a_1, \ldots, a_n) = t_-^{\mathbf{G}^-}(0 \wedge a_1, 0 \wedge -a_1, \ldots, 0 \wedge a_n, 0 \wedge -a_n).$$

The base case where t is a variable is trivial, and the inductive steps are as follows:

$$(-t)_+ := t_-, \qquad (t \wedge u)_+ := t_+ \wedge u_+, \qquad (t \vee u)_+ := t_+ \vee u_+,$$
$$(-t)_- := t_+, \qquad (t \wedge u)_- := t_- \vee u_-, \qquad (t \vee u)_- := t_- \wedge u_-,$$

$$(t + u)_+ := (t_+ + u_+) \ominus (t_- + u_-), \qquad (\square t)_+ := \square t,$$
$$(t + u)_- := (t_- + u_-) \ominus (t_+ + u_+), \qquad (\square t)_- := 0.$$

The step for $(t + u)_+$ uses the fact that, since $(0 \vee x) + (0 \wedge x) = x$,

$$\begin{aligned}
&((0 \wedge x) + (0 \wedge y)) \ominus ((0 \wedge -x) + (0 \wedge -y)) \\
&= 0 \wedge (-(0 \wedge -y) - (0 \wedge -x) + (0 \wedge x) + (0 \wedge y)) \\
&= 0 \wedge ((0 \vee x) + (0 \wedge x) + (0 \vee y) + (0 \wedge y)) = 0 \wedge (x + y)
\end{aligned}$$

Substituting $x \mapsto -x$ and $y \mapsto -y$ yields that $((0 \wedge -x) + (0 \wedge -y)) \ominus ((0 \wedge x) + (0 \wedge y)) = 0 \wedge (-x - y) = 0 \wedge -(x + y)$, establishing the step for $(t + u)_-$. The step for $\square$ uses the facts that $\square(0 \wedge x) = \square x = 0 \wedge \square x$ and $0 \wedge -\square x = 1$ for each a conucleus $\square$.

In each Abelian ℓ-group $x = y$ if and only if $0 \wedge x = 0 \wedge x$ and $0 \wedge -x = 0 \wedge -y$: right to left, $x = (0 \wedge x) + (0 \vee x) = (0 \wedge x) - (0 \wedge -x) = (0 \wedge y) - (0 \wedge -y) = (0 \wedge y) + (0 \vee y) = y$. Taking $\tau_-(t = u) := (t_+ = u_+) \,\&\, (t_- = u_-)$, for each Abelian ℓ-group equation $\varepsilon(x_1, \ldots, x_n)$ we have

$$\mathbf{G} \vDash \varepsilon(a_1, \ldots, a_n) \iff \mathbf{G}^- \vDash \tau_-(\phi)(0 \wedge a_1, 0 \wedge -a_1, \ldots, 0 \wedge a_n, 0 \wedge -a_n).$$

Letting τ_- commute with negations and conjunctions and disjunctions, we obtain the desired translation. $\square$

Remark 5.13. The paper [5] establishes a categorical equivalence between ℓ-groups and ℓ-group cones via the negative cone functor, using the description of categorical equivalence between varieties due to McKenzie [33]. In the same way, the translation given in the proof of Lemma 5.12 immediately yields an equivalence between the categories of (Abelian) ℓ-groups with a negative conucleus and of (Abelian) ℓ-group cones with a conucleus.

Theorem 5.14 (Generation for Abelian ℓ-groups with a negative conucleus)**.** *The variety of Abelian ℓ-groups with a negative conucleus is generated as follows:*

$$\mathbb{C}\mathsf{x}^{-}(\mathsf{AbLG}) = \mathbb{U}\mathbb{C}\mathsf{x}^{-}\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}).$$

Proof. This follows immediately from Theorem 5.9 and Lemma 5.12. ☐

Theorem 5.15 (Decidability for Abelian ℓ-groups with a negative conucleus)**.** *The universal theory of Abelian ℓ-groups with a negative conucleus is decidable.*

Proof. This follows immediately from Theorem 5.10 and Lemma 5.12. ☐

We can also easily derive generation and decidability results for the conuclear images (rather than conuclear expansions) of Abelian ℓ-group cones.

Theorem 5.16 (The Montagna–Tsinakis representation [37, 36])**.** *The conuclear images of Abelian ℓ-groups (Abelian ℓ-group cones) are precisely the (integral) cancellative commutative residuated lattices.*

Theorem 5.17 (Generation for cancellative ICRLs)**.** *The variety $\mathbb{C}(\mathsf{AbLG}^{-})$ of cancellative ICRLs is generated as follows:*

$$\mathbb{C}(\mathsf{AbLG}^{-}) = \mathbb{U}\mathbb{C}\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}).$$

Proof. This follows immediately from Theorem 5.14 and Lemma 2.7. ☐

Theorem 5.18 (Decidability for integral cancellative commutative RLs)**.** *The universal theory of integral cancellative commutative RLs is decidable.*

Proof. In view of Lemma 2.7, this follows immediately from the decidability of the universal theory of conuclear Abelian ℓ-group cones (Theorem 5.10) and the Montagna–Tsinakis representation (Theorem 5.16). ☐

6. Conuclear expansions of unit intervals

We now derive further corollaries of the generation and decidability results for conuclear Abelian ℓ-groups cones, namely generation and decidability results for conuclear MV-algebras (Theorems 6.8 and 6.10), which are precisely the unit intervals of conuclear Abelian ℓ-group cones (Theorem 6.7).

Unit intervals are a particular kind of nuclear images on integral residuated lattices. A *nucleus* on a residuated lattice $\mathbf{A}$ is a closure operator $\diamond$ which satisfies the inequality $\diamond x \cdot \diamond y \leq \diamond(x \cdot y)$. Its image

$$\mathbf{A}_{\diamond} := \{\diamond(a) \mid a \in \mathbf{A}\} = \{a \in \mathbf{A} \mid a = \diamond(a)\}$$

carries the structure of a residuated lattice which is a subalgebra of $\mathbf{A}$ with respect to $\wedge$, $\backslash$, $/$ and its join $\vee_{\diamond}$, product $\cdot_{\diamond}$, and unit $1_{\diamond}$ are defined as follows:

$$x \vee_{\diamond} y := \diamond(x \vee y), \qquad x \cdot_{\diamond} y := \diamond(x \cdot y), \qquad 1_{\diamond} := \diamond(1).$$

Residuated lattices of the form $\mathbf{A}_{\diamond}$ are called *nuclear images* of $\mathbf{A}$.

Given an integral residuated lattice $\mathbf{A}$ and $u \in \mathbf{A}$, the map $\diamond_u \colon x \mapsto u \vee x$ is a nucleus on $\mathbf{A}$ whose image is the interval $[u, 1]_{\mathbf{A}} := \mathbf{A}_{\diamond_u} = \{a \in \mathbf{A} \mid u \leq a \leq 1\}$. Note that $[u, 1]_{\mathbf{A}}$ is a *bounded* integral residuated lattice with bottom element u. We call such bounded integral residuated lattices *unit intervals* of $\mathbf{A}$.

Lemma 6.1. *Let $\Box$ be a conucleus and $\Diamond$ a nucleus on a residuated lattice $\mathbf{A}$ such that $\Diamond\Box x \leq \Box\Diamond x$. Then the restriction of $\Box$ to $\mathbf{A}_\Diamond$ is a conucleus on $\mathbf{A}_\Diamond$.*

Proof. For $a \in \mathbf{A}_\Diamond$ we have $\Box a \leq \Diamond\Box a \leq \Box\Diamond a = \Box a$, so indeed $\Box a \in \mathbf{A}_\Diamond$. Clearly the restriction of $\Box$ to $\mathbf{A}_\Diamond$ remains an order preserving map. Moreover, $1_\Diamond = \Diamond 1 = \Diamond\Box 1 \leq \Box\Diamond 1 = \Box(1_\Diamond) \leq 1_\Diamond$, so $\Box(1_\Diamond) = 1_\Diamond$. Finally, $\Box x \cdot_\Diamond \Box y = \Diamond(\Box x \cdot \Box y) \leq \Diamond\Box(x \cdot y) \leq \Box\Diamond(x \cdot y) = \Box(x \cdot_\Diamond y)$ for $x, y \in \mathbf{A}_\Diamond$. $\Box$

Accordingly, a nucleus $\Diamond$ on a *conuclear* residuated lattice $\langle \mathbf{A}, \Box \rangle$ is defined as a nucleus $\Diamond$ on $\mathbf{A}$ which satisfies the inequality $\Diamond\Box x \leq \Box\Diamond x$. The *nuclear image* of $\langle \mathbf{A}, \Box \rangle$ with respect to $\Diamond$ is $\mathbf{A}_\Diamond$ equipped with the restriction of $\Box$ to $\mathbf{A}_\Diamond$.

In particular, the condition $\Diamond_u \Box x \leq \Box\Diamond_u x$ is equivalent to $u = \Box u$. In that case, we write $\Box_u$ for the restriction of $\Box$ to $\mathbf{A}_{\Diamond_u}$ and we call bounded conuclear residuated lattices of the form $\langle \mathbf{A}_u, \Box_u \rangle$ *unit intervals* of $\langle \mathbf{A}, \Box \rangle$.

Given a (conuclear) integral residuated lattice $\mathbf{A}$, let $\mathbb{I}\mathrm{nt}(\mathbf{A})$ be the set of unit intervals of $\mathbf{A}$. Given a class K of (conuclear) integral residuated lattices, let $\mathbb{I}\mathrm{nt}(\mathsf{K})$ be the class of unit intervals of algebras in K. Keep in mind that $\mathbb{I}\mathrm{nt}(\mathsf{K})$ is a class of *bounded* (conuclear) integral residuated lattices.

The last item in the following lemma will not be needed in this paper, but it makes for a natural companion to the other items.

Lemma 6.2. *Let K be a class of (conuclear) integral residuated lattices. Then:*

(i) $\mathbb{I}\mathrm{nt}(\mathbb{I}(\mathsf{K})) = \mathbb{I}(\mathbb{I}\mathrm{nt}(\mathsf{K}))$.
(ii) $\mathbb{I}\mathrm{nt}(\mathbb{H}(\mathsf{K})) \subseteq \mathbb{H}(\mathbb{I}\mathrm{nt}(\mathsf{K}))$.
(iii) $\mathbb{I}\mathrm{nt}(\mathbb{S}(\mathsf{K})) \subseteq \mathbb{S}(\mathbb{I}\mathrm{nt}(\mathsf{K}))$.
(iv) $\mathbb{I}\mathrm{nt}(\mathbb{P}(\mathsf{K})) = \mathbb{P}(\mathbb{I}\mathrm{nt}(\mathsf{K}))$.
(v) $\mathbb{I}\mathrm{nt}(\mathbb{P}_{\mathsf{fin}}(\mathsf{K})) = \mathbb{P}_{\mathsf{fin}}(\mathbb{I}\mathrm{nt}(\mathsf{K}))$.
(vi) $\mathbb{I}\mathrm{nt}(\mathbb{P}_{\mathrm{U}}(\mathsf{K})) = \mathbb{P}_{\mathrm{U}}(\mathbb{I}\mathrm{nt}(\mathsf{K}))$.

If K is a class of (conuclear) integral commutative residuated lattices, then moreover:

(vii) $\mathbb{I}\mathrm{nt}(\mathbb{H}(\mathsf{K})) = \mathbb{H}(\mathbb{I}\mathrm{nt}(\mathsf{K}))$.

Proof. We omit the straightforward proofs for $\mathbb{I}$, $\mathbb{P}$, $\mathbb{P}_{\mathsf{fin}}$, and $\mathbb{P}_{\mathrm{U}}$.

(ii): consider $\mathbf{A} \in \mathsf{K}$, a surjective homomorphism $h \colon \mathbf{A} \twoheadrightarrow \mathbf{B}$, and $v \in \mathbf{B}$. There is some $u \in \mathbf{A}$ with $h(u) = v$. Consider the map $g \colon [u, 1]_{\mathbf{A}} \to [v, 1]_{\mathbf{B}}$ defined as the restriction of h to the interval $[u, 1]$. This map is surjective: given $y \in [v, 1]$, there is some $x \in \mathbf{A}$ with $h(x) = y$, so $h(x \vee u) = h(x) \vee h(u) = y \vee v = y$ and $x \vee u \in [u, 1]$. It is a homomorphism with respect to multiplication: $g(x \cdot^{[u,1]_{\mathbf{A}}} y) = g(u \vee (x \cdot y)) = h(u \vee (x \cdot y)) = h(u) \vee (h(x) \cdot h(y)) = v \vee (h(x) \cdot h(y)) = v \vee (g(x) \cdot g(y)) = g(x) \cdot^{[v,1]_{\mathbf{B}}} g(y)$. It is also a homomorphism with respect to all other residuated lattice operations, since these coincide in $[u, 1]_{\mathbf{A}}$ and $\mathbf{A}$, and in $[v, 1]_{\mathbf{B}}$ and $\mathbf{B}$.

(iii): consider $u \in \mathbf{A} \leq \mathbf{B} \in \mathsf{K}$. Then $[u, 1]_{\mathbf{A}} \leq [u, 1]_{\mathbf{B}} \in \mathbb{I}\mathrm{nt}(\mathsf{K})$, since $x \cdot^{[u,1]_{\mathbf{A}}} y = \Diamond_u^{\mathbf{A}}(x \cdot^{\mathbf{A}} y) = u \vee^{\mathbf{A}} (x \cdot^{\mathbf{A}} y) = u \vee^{\mathbf{B}} (x \cdot^{\mathbf{B}} y) = \Diamond_u^{\mathbf{B}}(x \cdot^{\mathbf{B}} y) = x \cdot^{[u,1]_{\mathbf{B}}} y$ and all other operations in $\mathbf{A}$ coincides with those of $\mathbf{A}$ and therefore those of $\mathbf{B}$.

(vii): it remains to show that $\mathbb{H}(\mathbb{I}\mathrm{nt}(\mathsf{K})) \subseteq \mathbb{I}\mathrm{nt}(\mathbb{H}(\mathsf{K}))$ if $\mathsf{K} \subseteq \mathsf{ICRL}$. We use the correspondence between the homomorphic images of $\mathbf{A} \in \mathsf{ICRL}$ and the (multiplicative) filters of $\mathbf{A}$, that is, non-empty upsets F closed under binary products. The filters of $\mathbf{A}$ are in bijective correspondence with the congruences of $\mathbf{A}$: the filter F induces the congruence θ_F such that $\langle a, b \rangle \in \theta_F$ if and only if $a \backslash b, b \backslash a \in F$. We write a/F and $\mathbf{A}/F$ for a/θ_F and $\mathbf{A}/F$.

Consider an algebra $\mathbf{A} \in \mathsf{K}$, an element $u \in \mathbf{A}$, and a filter F of $[u, 1]_{\mathbf{A}}$. Let G be the filter of $\mathbf{A}$ generated by F. Then $F = G \cap [u, 1]_{\mathbf{A}}$: clearly $F \subseteq G \cap [u, 1]_{\mathbf{A}}$, and conversely if $a \in G \cap [u, 1]_{\mathbf{A}}$, then there are $f_1, \dots, f_n \in F$ such that $f_1 \cdot^{\mathbf{A}} \dots \cdot^{\mathbf{A}} f_n \leq a$, so $u \vee (f_1 \cdot^{\mathbf{A}} \dots \cdot^{\mathbf{A}} f_n) \leq a$ and $f_1 \cdot^{[u,1]_{\mathbf{A}}} \dots \cdot^{[u,1]_{\mathbf{A}}} f_n \leq a$, hence $a \in F$.

It suffices to show that $[u/F, 1]_{\mathbf{A}/F} = [u/G, 1/G]_{\mathbf{A}/G}$. But the equality $a/F = b/F$ for $a, b \in [u, 1]_{\mathbf{A}}$ holds if and only if $a \backslash b, b \backslash a \in F = G \cap [u, 1]_{\mathbf{A}}$, which is equivalent to $a \backslash b, b \backslash a \in G$, and therefore to $a/G = b/G$. The underlying sets of $[u/F, 1]_{\mathbf{A}/F}$ and $[u/G, 1/G]_{\mathbf{A}/G}$ thus coincide. To prove that products coincide, consider $a, b \in [u, 1]_{\mathbf{A}}$. The product of a/F and b/F in $[u, 1]_{\mathbf{A}/F}$ is $(u \vee^{\mathbf{A}} (a \cdot^{\mathbf{A}} b))/F$. The product of $a/G = a/F$ and $b/G = b/F$ in $[u/G, 1/G]_{\mathbf{A}/G}$ is $(u/G) \vee^{\mathbf{A}/G} ((a/G) \cdot^{\mathbf{A}/G} (b/G)) = (u \vee^{\mathbf{A}} (a \cdot^{\mathbf{A}} b))/G = (u \vee^{\mathbf{A}} (a \cdot^{\mathbf{A}} b))/F$. A similar but simpler argument shows that the other residuated lattice operations also coincide in the two algebras. $\square$

The next two lemmas and theorem were in effect proved in an unpublished note by Young [48]. We reproduce his proof below. What Young in fact proved is that conuclear images of MV-algebras are precisely the unit intervals of integral cancellative commutative residuated lattices. However, his proofs apply equally well in a more general setting. In the following, a *meet preserving* nucleus is a nucleus $\Diamond$ such that $\Diamond(x \wedge y) = \Diamond x \wedge \Diamond y$.

Lemma 6.3 ([48])**.** *Let $\mathbf{A}$ be a residuated lattice, $\Diamond$ a meet preserving nucleus on $\mathbf{A}$, and $\Box$ a conucleus on $\mathbf{A}_{\Diamond}$. Then $\Box$ extends to a conucleus $\overline{\Box} := \Box\Diamond x \wedge x$ on $\mathbf{A}$ such that $\Diamond$ is a nucleus on $\langle \mathbf{A}, \overline{\Box} \rangle$ and $\Box = \overline{\Box}_{\Diamond}$. Moreover, $\overline{\Diamond} := \Box\Diamond x$ is a nucleus on $\mathbf{A}_{\overline{\Box}}$ such that $(\mathbf{A}_{\Diamond})_{\Box} = (\mathbf{A}_{\overline{\Box}})_{\overline{\Diamond}}$. If $\Diamond = \Diamond_u$ for some $u \in \mathbf{A}$, then $\overline{\Diamond} = x \vee_{\overline{\Box}} u$.*

Proof. We show that $\overline{\Box} := \Box\Diamond x \wedge x$ is a conucleus on $\mathbf{A}$. It is clearly an order preserving map because $\Box$ and $\Diamond$ are order preserving. The inequality $\overline{\Box} x \leq x$ holds by definition. Moreover,

$$\overline{\Box}(\overline{\Box} x) = \Box(\Diamond\overline{\Box} x) \wedge \overline{\Box} x = \Box(\Diamond(\Box\Diamond x \wedge x)) \wedge \Box\Diamond x \wedge x.$$

But $\Box\Diamond x \leq \Diamond\Box\Diamond x \wedge \Diamond(x) = \Diamond(\Box\Diamond x \wedge x)$, therefore $\Box\Diamond x \leq \Box\Diamond(\Box\Diamond x \wedge x)$. Thus, $\overline{\Box}\overline{\Box} x = \overline{\Box} x$. Finally, $\overline{\Box} x \cdot \overline{\Box} y \leq \overline{\Box}(x \cdot y)$:

$$\begin{aligned}
\overline{\Box} x \cdot \overline{\Box} y &= (\Box\Diamond x \wedge x) \cdot (\Box\Diamond y \wedge y) \\
&\leq (\Box\Diamond x \cdot \Box\Diamond y) \wedge xy \\
&\leq \Diamond(\Box\Diamond x \cdot \Box\Diamond y) \wedge xy = (\Box\Diamond x \cdot_{\Diamond} \Box\Diamond y) \wedge xy \\
&\leq \Box(\Diamond x \cdot_{\Diamond} \Diamond y) \wedge xy = \Box\Diamond(\Diamond x \cdot \Diamond y) \wedge xy \\
&\leq \Box\Diamond\Diamond(xy) \wedge xy = \Box\Diamond(xy) \wedge xy = \overline{\Box}(xy).
\end{aligned}$$

This proves that $\overline{\Box}$ is a conucleus on $\mathbf{A}$. For $x \in \mathbf{A}_{\Diamond}$ clearly $\overline{\Box} x = \Box\Diamond x \wedge x = \Box x \wedge x = \Box x$, so $\overline{\Box}$ is indeed an extension of $\Box$ to $\mathbf{A}$.

Let $\overline{\Diamond} x := \overline{\Box}\Diamond x$. Then $\overline{\Diamond} x = \Box\Diamond(\Diamond x) \wedge \Diamond x = \Box\Diamond x \wedge \Diamond x = \Box\Diamond x$. We show that $\overline{\Diamond}$ is a nucleus on $\mathbf{A}_{\overline{\Box}}$. It is clearly an order preserving map because $\overline{\Box}$ and $\Diamond$ are order preserving. The inequality $x \leq \overline{\Diamond} x$ holds for $x \in \mathbf{A}_{\overline{\Box}}$ because $x \leq \Diamond x$ and therefore $x = \overline{\Box} x \leq \overline{\Box}\Diamond x = \overline{\Diamond} x$. Moreover,

$$\overline{\Diamond}(\overline{\Diamond} x) = \overline{\Box}\Diamond\overline{\Box}\Diamond x \leq \overline{\Box}\Diamond\Diamond x = \overline{\Box}\Diamond x = \overline{\Diamond} x,$$

and

$$\overline{\Diamond} x \cdot \overline{\Diamond} y = \overline{\Box}\Diamond x \cdot \overline{\Box}\Diamond y \leq \overline{\Box}(\Diamond x \cdot \Diamond y) \leq \overline{\Box}\Diamond(x \cdot y) = \overline{\Diamond}(x \cdot y).$$

Since $\overline{\Diamond}x \in \mathbf{A}_{\overline{\Box}}$ by definition, this proves that $\overline{\Diamond}$ is a nucleus on $\mathbf{A}_{\overline{\Box}}$. To prove that it is a nucleus on $\langle \mathbf{A}, \overline{\Box} \rangle$, we need to show that $\overline{\Diamond}\overline{\Box}x \leq \overline{\Box}\overline{\Diamond}x$. But indeed $\overline{\Diamond}\overline{\Box}x = \overline{\Diamond}(\Box\Diamond x \wedge x) = \Box\Diamond(\Box\Diamond x \wedge x) \leq \Box\Diamond\Box\Diamond x \wedge \Box\Diamond x = \overline{\Box}(\Box\Diamond x) = \overline{\Box}(\overline{\Diamond}x)$.

We now show that $(\mathbf{A}_{\Diamond})_{\Box} = (\mathbf{A}_{\overline{\Box}})_{\overline{\Diamond}}$. Given $x = \Diamond x = \Box x$, we have $\overline{\Box}x = \Box\Diamond x \wedge x = x$ and $\overline{\Diamond}x = \Diamond x = x$. Conversely, given $x = \overline{\Box}x = \overline{\Diamond}x$, we have $x = \Diamond x = \Box\Diamond x = \Box x$. The underlying sets of the algebras $(\mathbf{A}_{\Diamond})_{\Box}$ and $(\mathbf{A}_{\overline{\Box}})_{\overline{\Diamond}}$ are therefore equal. Moreover, in both algebras the order is the restriction of the order of $\mathbf{A}$ and multiplication is the operation $\Diamond(x \cdot y)$, so the two algebras are equal.

Finally, suppose that $\Diamond x = x \vee^{\mathbf{A}} u$ for some $u \in \mathbf{A}$. Then $u \in \mathbf{A}_{\overline{\Box}}$, since $\overline{\Box}u = \Box\Diamond u \wedge u = \Box u \wedge u = \Box u = u$, where the last equality holds because $\Box$ is an interior operator on $\mathbf{A}_{\Diamond}$ and u is the least element of $\mathbf{A}_{\Diamond}$. Moreover, $\overline{\Diamond}x = x \vee^{\mathbf{A}_{\overline{\Box}}} u$, since $\overline{\Diamond}x = \overline{\Box}\Diamond x = \overline{\Box}(x \vee u) = x \vee^{\mathbf{A}}_{\overline{\Box}} u$. □

Lemma 6.4 ([48]). *For each class* K *of integral residuated lattices*

$$\mathbb{I}\mathrm{nt}(\mathbb{C}\mathrm{x}(\mathsf{K})) \subseteq \mathbb{C}\mathrm{x}(\mathbb{I}\mathrm{nt}(\mathsf{K})), \qquad \mathbb{I}\mathrm{nt}(\mathbb{C}(\mathsf{K})) \subseteq \mathbb{C}(\mathbb{I}\mathrm{nt}(\mathsf{K})).$$

Proof. Consider an integral residuated lattice $\mathbf{A}$, a conucleus $\Box$ on $\mathbf{A}$, and an element $u \in \mathbf{A}_{\Box}$. The conucleus $\Box$ restricts to an interior operator $\Box_u$ on $[u, 1]_{\mathbf{A}}$, since $u \leq a$ implies $u = \Box u \leq \Box a$. The image of $\Box_u$ is the same as the image of $\Box$ restricted to $[u, 1]_{\mathbf{A}}$. Thus the interval $\langle [u, 1]_{\mathbf{A}}, \Box_u \rangle \vDash \phi$ of the conuclear residuated lattice $\langle \mathbf{A}, \Box \rangle$ is a conuclear expansion of the interval $[u, 1]_{\mathbf{A}}$ of $\mathbf{A}$. □

Theorem 6.5 ([48]). *For each class* K *of distributive integral residuated lattices*

$$\mathbb{I}\mathrm{nt}(\mathbb{C}\mathrm{x}(\mathsf{K})) = \mathbb{C}\mathrm{x}(\mathbb{I}\mathrm{nt}(\mathsf{K})), \qquad \mathbb{I}\mathrm{nt}(\mathbb{C}(\mathsf{K})) = \mathbb{C}(\mathbb{I}\mathrm{nt}(\mathsf{K})).$$

Proof. This follows immediately from Lemma 6.3 and 6.4, since the nucleus $\Diamond_u x := x \vee u$ is meet preserving in each distributive integral residuated lattice. □

The above theorem now allows us to upgrade Mundici's representation of MV-algebras as intervals of Abelian ℓ-group cones [38] to conuclear MV-algebras.

Theorem 6.6 (The Mundici representation of MV-algebras [38]).
MV-algebras (totally ordered MV-algebras) are precisely the unit intervals of Abelian ℓ-group cones (totally ordered Abelian ℓ-group cones).

Theorem 6.7 (The Young representation of conuclear MV-algebras [48]).
Conuclear MV-algebras (totally ordered conuclear MV-algebras) are precisely the unit intervals of (totally ordered) conuclear Abelian ℓ-group cones. Consequently, the conuclear images of MV-algebras are precisely the unit intervals of cancellative ICRL*s.*

Proof. The first claim follows immediately from the Mundici representation of MV-algebras (Theorem 6.6) and Theorem 6.5, while the second claim follows from the first claim plus the Montagna–Tsinakis representation of cancellative ICRLs (Theorem 5.16). □

Let $\mathsf{MV}_{\mathrm{fin}}$ denote the class of finite MV-algebras. Since $\mathbb{I}\mathrm{nt}(\mathbb{Z}^-)$ is the class of all finite totally ordered MV-algebras, $\mathsf{MV}_{\mathrm{fin}} = \mathbb{I}\mathbb{P}_{\mathrm{fin}}(\mathbb{I}\mathrm{nt}(\mathbb{Z}^-))$ [17, Proposition 3.6.5].

Theorem 6.8 (Generation for conuclear MV-algebras).
The variety of conuclear MV-algebras is generated as follows:

$$\mathbb{C}\mathrm{x}(\mathsf{MV}) = \mathbb{U}\mathbb{C}\mathrm{x}\mathbb{P}_{\mathrm{fin}}(\mathbb{I}\mathrm{nt}(\mathbb{Z}^-)) = \mathbb{U}\mathbb{C}\mathrm{x}(\mathsf{MV}_{\mathrm{fin}}).$$

In particular, it has the Finite Embeddability Property.

Proof. Theorems 5.9 and 6.7 yield the equalities $\mathbb{C}\mathsf{x}(\mathsf{MV}) = \mathbb{I}\mathrm{nt}(\mathbb{C}\mathsf{x}(\mathsf{AbLG}^-)) = \mathbb{I}\mathrm{nt}(\mathbb{U}\mathbb{C}\mathsf{x}\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-))$, so $\mathbb{C}\mathsf{x}(\mathsf{MV}) \subseteq \mathbb{U}\mathbb{C}\mathsf{x}\mathbb{P}_{\mathrm{fin}}(\mathbb{I}\mathrm{nt}(\mathsf{K}))$ holds by Lemma 6.2 and Theorem 6.5. Conversely, $\mathbb{I}\mathrm{nt}(\mathbb{Z}^-) \subseteq \mathsf{MV}$, so $\mathbb{U}\mathbb{C}\mathsf{x}\mathbb{P}_{\mathrm{fin}}(\mathbb{I}\mathrm{nt}(\mathbb{Z}^-)) \subseteq \mathbb{U}\mathbb{C}\mathsf{x}(\mathsf{MV}) = \mathbb{C}\mathsf{x}(\mathsf{MV})$. □

Lemma 6.9. *Let* K *be a class of (conuclear) integral residuated lattices. If the universal theory of* K *is decidable, then so is the universal theory of* $\mathbb{I}\mathrm{nt}(\mathsf{K})$.

Proof. Given the definition of the operations of $[u, 1]_{\mathbf{A}}$ in terms of the operations of the (conuclear) integral residuated lattice $\mathbf{A}$, it is straightforward to define a translation τ_{int} from terms in n variables in the signature of (conuclear) bounded integral residuated lattices to terms in $n+1$ variables in the signature of (conuclear) integral residuated lattices, the last variable being the translation of the constant $\bot$, and to extend this translation to universal sentences in such a way that $\mathbb{I}\mathrm{nt}(\mathbf{A}) \vDash \phi$ if and only if $\mathbf{A} \vDash \tau_{\mathrm{int}}(\phi)$. □

Theorem 6.10 (Decidability for conuclear MV-algebras)**.**
The universal theory of conuclear MV-algebras is decidable.

Proof. This follows immediately from Theorems 5.10 and 6.7 and Lemma 6.9. □

Describing the class $\mathbb{C}(\mathsf{MV})$ remains an open problem [36, Problem 10].

7. Meet preserving conuclei

The central observation on which the paper has rested thus far is that given a finite subset X of an integral residuated lattice $\mathbf{A}$ and a coinitial subset $S \subseteq X$, the sub-sℓ-monoid $\langle S \rangle^{\mathbf{A}}$ generated by S is the image of a conucleus $\Box_S$ on the subalgebra $\mathrm{Sg}^{\mathbf{A}} X$ of $\mathbf{A}$. In this section, we shall restrict our attention to meet preserving conuclei. This will involve replacing the sub-sℓ-monoid $\langle S \rangle^{\mathbf{A}}$ by the sub-ℓ-monoid $\langle\!\langle S \rangle\!\rangle^{\mathbf{A}}$. We shall go over the results proved thus far, section by section, and state how they need to be modified to account for this. Most results will be entirely straightforward to modify, and in those case we omit the relevant proofs to avoid duplicating the entire content of the paper thus far.

An *ℓ-monoid* is, for the purposes of this paper, an algebra $\langle A, \wedge, \vee, \cdot, 1 \rangle$ such that $\langle A, \wedge, \vee \rangle$ is a lattice and $\langle A, \vee, \cdot, 1 \rangle$ is an sℓ-monoid. A *$\wedge$-pomonoid* is an algebra $\langle A, \wedge, \cdot, 1 \rangle$ such that $\langle A, \wedge \rangle$ is a meet semilattice, $\langle A, \cdot, 1 \rangle$ is a monoid, and moreover the following equations are satisfied:

$$x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z), \qquad (x \wedge y) \cdot z = (x \cdot z) \wedge (y \cdot z).$$

A *distributive ℓ-monoid* is an algebra $\langle A, \wedge, \vee, \cdot, 1 \rangle$ such that $\langle A, \wedge, \vee \rangle$ is a distributive lattice and binary products distribute over both binary meets and joins, i.e. $\langle A, \vee, \cdot, 1 \rangle$ is an sℓ-monoid and $\langle A, \wedge, \cdot, 1 \rangle$ is a $\wedge$-pomonoid. (The asymmetry in the terminology reflects the fact that in this section, $\wedge$-pomonoids will play the same role that pomonoids played in Section 3.)

A *meet preserving conucleus* or a *$\wedge$-conucleus* for short on a residuated lattice $\mathbf{A}$ is a conucleus $\Box$ on $\mathbf{A}$ which satisfies the equation $\Box(x \wedge y) = \Box x \wedge \Box y$. Equivalently, it is a conucleus $\Box$ whose image is closed under binary meets, i.e. $\mathbf{A}_\Box$ is a sub-ℓ-monoid of $\mathbf{A}$: being meet preserving implies that $\mathbf{A}_\Box$ is closed under binary meets, and if $\mathbf{A}_\Box$ is closed under binary meets, then $\Box x \wedge \Box y \in \mathbf{A}_\Box$, so $\Box x \wedge \Box(y = \Box(\Box x \wedge \Box y) \leq \Box(x \wedge y) \leq \Box x \wedge \Box y$.

Given a class K of residuated lattices, $\mathbb{C}_\wedge(\mathsf{K})$ denotes the class of all $\wedge$-conuclear images of algebras in K and $\mathbb{C}\mathrm{x}_\wedge(\mathsf{K})$ denotes the class of all $\wedge$-conuclear expansions of algebras in K. Because the identity map is a $\wedge$-conucleus, $\mathsf{K} \subseteq \mathbb{C}_\wedge(\mathsf{K})$ and each residuated lattice is the reduct of some $\wedge$-conuclear residuated lattice.

Recall that we call a residuated lattice fully distributive if its division-free reduct is a distributive ℓ-monoid. The variety of fully distributive IRLs (ICRLs) will be denoted by FdIRL (FdICRL). Because these varieties are axiomatized relative to IRL by equations in the signature of ℓ-monoids, they are closed under meet preserving conuclei. That is, $\mathbb{C}_\wedge(\mathsf{FdIRL}) = \mathsf{FdIRL}$ and $\mathbb{C}_\wedge(\mathsf{FdICRL}) = \mathsf{FdICRL}$.

Given a fully distributive $\mathbf{A} \in \mathsf{IRL}$ and a finite set $S \subseteq \mathbf{A}$, let

$$\langle\!\langle S \rangle\!\rangle^{\mathbf{A}} := \text{sub-}\ell\text{-monoid of } \mathbf{A} \text{ generated by } S.$$

That is, $\langle\!\langle S \rangle\!\rangle^{\mathbf{A}}$ consists of non-empty finite joins of non-empty finite meets of finite products of elements of S. In other words, $\langle\!\langle S \rangle\!\rangle^{\mathbf{A}}$ is the join semilattice generated by the $\wedge$-pomonoid generated by S.

We now go through the results of Section 3, replacing conuclei by $\wedge$-conuclei, $\langle S \rangle^{\mathbf{A}}$ by $\langle\!\langle S \rangle\!\rangle^{\mathbf{A}}$, and π-embeddings by what we call $\wedge\pi$-embeddings. This is an entirely mechanical task. We therefore omit the proofs except for the next lemma, which replaces Lemma 2.12.

Lemma 7.1. *Each finitely generated integral distributive ℓ-monoid satisfies the ascending chain condition.*

Proof. Let $\mathbf{A}$ be an integral distributive ℓ-monoid generated by a finite set $X \subseteq \mathbf{A}$. Let $\mathbf{M}$ be the submonoid of $\mathbf{A}$ generated by X. A double application of Higman's Lemma (Lemma 2.11) shows that the meet subsemilattice of $\mathbf{A}$ generated by $\mathbf{M}$ is well partially ordered. Lemma 2.12 then shows that the join subsemilattice generated by the meet subsemilattice generated by $\mathbf{A}$ satisfies the ascending chain condition. But this join subsemilattice is $\mathbf{A}$. □

Lemma 7.2 (The conuclear integral residuated lattice $\langle \mathrm{Sg}^{\mathbf{A}} X, \Box^{\wedge}_{S} \rangle$)**.**
Consider a fully distributive $\mathbf{A} \in \mathsf{IRL}$, a set $X \subseteq \mathbf{A}$, and a finite coinitial $S \subseteq X$. Then $\langle\!\langle S \rangle\!\rangle^{\mathbf{A}}$ is the image of a meet preserving conucleus $\Box^{\wedge}_{S}$ on $\mathrm{Sg}^{\mathbf{A}} X$.

Lemma 7.3 (Finite partial subalgebras of $\wedge$-conuclear IRLs)**.**
Each finite partial subalgebra of a $\wedge$-conuclear integral residuated lattice $\langle \mathbf{A}, \Box \rangle$ is a restriction of $\langle \mathrm{Sg}^{\mathbf{A}} X, \Box^{\wedge}_{S} \rangle|_X$ for some finite $X \subseteq \mathbf{A}$ and coinitial $S \subseteq X$.

Lemma 7.4. *Consider $\mathsf{K} \subseteq \mathsf{FdIRL}$. If $\mathbb{C}\mathrm{x}_\wedge(\mathsf{K})$ has the FEP, then so does $\mathbb{C}_\wedge(\mathsf{K})$.*

Theorem 7.5 (FEP for $\wedge$-conuclear locally finite K-algebras)**.**
Let $\mathsf{K} \subseteq \mathsf{IRL}$ be a locally finite universal class. Then $\mathbb{C}\mathrm{x}_\wedge(\mathsf{K})$ and $\mathbb{C}_\wedge(\mathsf{K})$ have the FEP.

Consider a fully distributive $\mathbf{A} \in \mathsf{IRL}$, finite $X \subseteq \mathbf{A}$, and coinitial $S \subseteq X$. An embedding $\iota\colon \mathbf{A}|_X \hookrightarrow \mathbf{B}$ into a residuated lattice $\mathbf{B}$ will be called a *$\wedge\pi$-embedding* if for all words $w_1, \ldots, w_n \in \text{Word}\, X$ and all $a \in X$

$$w_1^{\mathbf{A}} \wedge \cdots \wedge w_n^{\mathbf{A}} \leq^{\mathbf{A}} a \implies \iota(w_1)^{\mathbf{B}} \wedge \cdots \wedge \iota(w_n)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a).$$

Lemma 7.6 ($\mathbb{U}\mathbb{C}\mathrm{x}_\wedge\mathbb{S}$ and $\wedge\pi$-embeddings)**.**
Suppose that each finite partial subalgebra of $\mathbf{A} \in \mathsf{FdIRL}$ has a $\wedge\pi$-embedding into some algebra in $\mathsf{K} \subseteq \mathsf{FdIRL}$. Then $\mathbb{C}\mathrm{x}_\wedge\mathbb{S}(\mathbf{A}) \subseteq \mathbb{U}\mathbb{C}\mathrm{x}_\wedge\mathbb{S}(\mathsf{K})$.

A fully distributive IRL has a locally finite monoid reduct if and only if it has a locally finite distributive ℓ-monoid reduct.

Theorem 7.7 ($\mathbb{U}\mathbb{C}\mathsf{x}_\wedge = \mathbb{C}\mathsf{x}_\wedge\mathbb{U}$ in the case of locally finite monoid reducts)**.**
Consider $\mathsf{K} = \mathbb{S}(\mathsf{K}) \subseteq \mathsf{FdIRL}$ *such that* $\mathbb{U}(\mathsf{K})$ *has locally finite monoid reducts. Then*

$$\mathbb{C}\mathsf{x}_\wedge\mathbb{U}(\mathsf{K}) = \mathbb{U}\mathbb{C}\mathsf{x}_\wedge(\mathsf{K}).$$

Theorem 7.8 (FEP for conuclear FdIRLs with locally finite monoid reducts)**.**
Let K *be a universal class of* FdIRL*s with locally finite monoid reducts. If* K *has the FEP, then so does the universal class* $\mathbb{C}\mathsf{x}_\wedge(\mathsf{K})$ *of their* $\wedge$*-conuclear expansions.*

Corollary 7.9 (FEP for n-potent $\wedge$-conuclear FdICRLs)**.**
The variety of n-potent $\wedge$*-conuclear* FdICRL*s has the FEP.*

We do not have an analogue of Lemma 3.12 of $\wedge$-conuclei. However, we have an analogue of Lemma 3.13.

Lemma 7.10. *Consider* $\mathbf{A} \in \mathsf{IRL}$ *with computable primitive operations. Each pair of decision procedures for the conditions* $\langle\!\langle s_1, \dots, s_n \rangle\!\rangle^{\mathbf{A}} \cap {\downarrow} a = \emptyset$ *for* $a, s_1, \dots, s_n \in \mathbf{A}$ *and* $\langle\!\langle s_1, \dots, s_n \rangle\!\rangle^{\mathbf{A}} \cap ({\downarrow} a - {\downarrow} b) = \emptyset$ *for* $b \leq a$ *in* $\mathbf{A}$ *and* $s_1, \dots, s_n \in \mathbf{A}$ *yields a computation procedure for the partial map* $\langle S, a \rangle \mapsto \Box^{\wedge}_{S} a$*, where* $S := \{s_1, \dots, s_n\} \subseteq \mathbf{A}$*.*

Next, we move to Section 4. Here the task of modifying our previous results necessitates more substantial changes to some proofs. In particular, ω-embeddings need to be replaced by what we call n-$\wedge\omega$-embeddings, which are parametrized by $n \in \mathbb{N}$. We again omit those proofs whose modification is straightforward.

Given a pomonoid $\mathbf{M}$, let $\mathrm{Down}_\omega\, \mathbf{M}$ denote the sℓ-monoid of finitely generated downsets of $\mathbf{M}$ ordered by inclusion. Binary joins are unions in $\mathrm{Down}_\omega\, \mathbf{M}$, the monoidal unit is ${\downarrow} 1^{\mathbf{M}}$, and products are defined as

$$X * Y := {\downarrow}\{x \cdot y \mid x \in X \text{ and } y \in Y\}.$$

The map $a \mapsto {\downarrow} a$ embeds $\mathbf{M}$ into $\mathrm{Down}_\omega\, \mathbf{M}$ as a subpomonoid.

Applying this construction to the pomonoid $\mathbf{M} := \mathrm{Multi}\, X$ yields the sℓ-monoid $\mathrm{Multi}_\vee\, X := \mathrm{Down}_\omega\, \mathrm{Multi}\, X$. The elements of $\mathrm{Multi}_\vee\, X$ are thus finitely generated sets of multisets which are closed under taking submultisets.

Given an integral residuated lattice $\mathbf{A}$, a finite set $X \subseteq \mathbf{A}$, and a finitely generated downset $W := {\downarrow}\{w_1, \dots, w_n\} \in \mathrm{Multi}_\vee\, X$, we use the notation

$$W^{\mathbf{A}} := w_1^{\mathbf{A}} \wedge \dots \wedge w_n^{\mathbf{A}}.$$

Lemma 7.11 (Higman's Lemma for sets of multisets)**.**
Let X *be a finite set. Then each upset of* $\mathrm{Multi}_\vee\, X$ *is finitely generated.*

Proof. By Higman's Lemma (Lemma 2.11), if a pomonoid $\mathbf{M}$ is dually integral (the monoidal unit is the bottom element) and generated by a well partially ordered set, then $\mathbf{M}$ itself is well partially ordered. Applying this lemma once to the submonoid of $\mathrm{Multi}_\vee\, X$ generated by the set $\{{\downarrow}[x] \mid x \in X\} \subseteq \mathrm{Multi}_\vee\, X$ shows that $\{{\downarrow} w \mid w \in \mathrm{Multi}\, X\}$ is a well partially ordered subset of $\mathrm{Multi}_\vee\, X$. Applying the lemma again to the join subsemilattice of $\mathrm{Multi}_\vee\, X$ generated by this submonoid now yields the desired conclusion. □

Downsets of $\mathrm{Multi}_\vee\, X$ need not be finitely generated. However, we show that they are still finite unions of downsets of a special form. Let us call a downset D

of $\text{Multi}_\vee X$ *quasi-principal* if there are $u_1, v_1, \dots, u_k, v_k \in \text{Multi } X$ such that

$$W \in D \iff W \subseteq {\downarrow}(u_1 \oplus mv_1) \cup \dots \cup {\downarrow}(u_k \oplus mv_k) \text{ for some } m \in \mathbb{N}.$$

For comparison, observe that a downset D of $\text{Multi}_\vee X$ is principal if there are $u_1, \dots, u_k \in \text{Multi } X$ such that

$$W \in D \iff W \subseteq {\downarrow} u_1 \cup \dots \cup {\downarrow} u_k.$$

A downset of $\text{Multi}_\vee X$ will be called *quasi-finitely generated* if it is a finite union of quasi-principal downsets.

Lemma 7.12 (Downsets of $\text{Multi}_\vee X$)**.**
Let S be a finite set. Then each downset of $\text{Multi}_\vee X$ *is a quasi-finitely generated.*

Proof. Consider a downset D of $\text{Multi}_\vee X$. If $D = \text{Multi}_\vee X$, the claim holds for $k := 1$ if we take $u_1 := \emptyset$ and we take $v_1 := {\downarrow} w_1$, where w_1 is the multiset containing exactly 1 occurrence of each element of X. If $D = \emptyset$, the claim holds trivially, since D is then the union of an empty family of quasi-principal downsets. We may therefore assume that $\emptyset \subsetneq D \subsetneq \text{Multi}_\vee X$.

Consider the non-empty upset $U := \text{Multi}_\vee X - D$ of $\text{Multi } X$. By Higman's Lemma (Lemma 2.11) the condition $W \in U$ is equivalent to a non-empty finite disjunction of conditions of the form $W' \subseteq W$ for some $W' \in \text{Multi}_\vee X$. Because D is non-empty, $U \subsetneq \text{Multi}_\vee X$, so each of these $W' \in \text{Multi}_\vee X$ is non-empty. It thus has the form $W' := {\downarrow} w_1 \cup \dots \cup {\downarrow} w_k$ for some $k \geq 1$ and $w_1, \dots, w_k \in \text{Multi } X$. Consequently, $W' \subseteq W$ if and only if ${\downarrow} w_i \subseteq W$ for each $i \in \{1, \dots, k\}$, i.e. if and only if $w_i \in W$ for each $i \in \{1, \dots, k\}$.

Negating this non-empty finite disjunction of non-empty finite conjunctions of conditions of the form $w_i \in W$ and transforming the negation into disjunctive normal form yields that the condition $W \in D$ is equivalent to a non-empty finite disjunction of non-empty finite conjunctions of conditions of the form $w_i \notin W$. It therefore suffices to show that a downset D of $\text{Multi}_\vee X$ is quasi-principal if there are $w_1, \dots, w_k \in \text{Multi } X$ with $k \geq 1$ such that

$$W \in D \iff w_1 \notin W \text{ and } \dots \text{ and } w_k \notin W.$$

Take

$$W_D := \{w \in \text{Multi } X \mid w_1 \not\sqsubseteq w \text{ and } \dots \text{ and } w_k \not\sqsubseteq w\}.$$

Then for each $W \in \text{Multi}_\vee X$

$$W \in D \iff W \subseteq W_D.$$

Note that it does not follow that D is a principal downset of $\text{Multi}_\vee X$, since the downset W_D need not be finitely generated.

By Lemma 4.2 the downset W_D of $\text{Multi } X$ is quasi-finitely generated. That is, there are $u_1, v_1, \dots, u_l, v_l \in \text{Multi } X$ such that

$$w \in W_D \iff w \sqsubseteq u_1 \oplus mv_1 \text{ or } \dots \text{ or } w \sqsubseteq u_l \oplus mv_l \text{ for some } m \in \mathbb{N},$$

or in other words

$$w \in W_D \iff w \in {\downarrow}(u_1 \oplus mv_1) \cup \dots \cup {\downarrow}(u_l \oplus mv_l) \text{ for some } m \in \mathbb{N},$$

The elements of D are then precisely the finitely generated downsets of W_D, so they are precisely the sets $W \in \text{Multi}_\vee X$ such that

$$W \subseteq {\downarrow}(u_1 \oplus mv_1) \cup \dots \cup {\downarrow}(u_l \oplus mv_l) \text{ for some } m \in \mathbb{N},$$

In other words, D is quasi-principal. $\square$

We shall use the abbreviation

$$a_1 \cdot b_1^{\omega} \wedge \cdots \wedge a_n \cdot b_n^{\omega} \nleq^{\mathbf{A}} c$$

for the claim that $a_1 \cdot b_1^{p} \wedge \cdots \wedge a_n \cdot b_n^{p} \nleq^{\mathbf{A}} c$ for all $p \in \mathbb{N}$.

Consider $\mathbf{A} \in \mathsf{FdICRL}$, a finite set $X \subseteq \mathbf{A}$, and $n \geq 1$. An embedding ι of $\mathbf{A}|_X$ into $\mathbf{B} \in \mathsf{FdICRL}$ will be called an *n-$\wedge\omega$-embedding* if for all $a_1, b_1, \ldots, a_n, b_n, c \in \mathbf{A}$

$$a_1 \cdot b_1^{\omega} \wedge \cdots \wedge a_n \cdot b_n^{\omega} \nleq^{\mathbf{A}} c \implies \iota(a_1) \cdot \iota(b_1)^{\omega} \wedge \cdots \wedge \iota(a_n) \cdot \iota(b_n)^{\omega} \nleq^{\mathbf{B}} \iota(c).$$

Lemma 7.13. *Consider totally ordered $\mathbf{A}, \mathbf{B} \in \mathsf{ICRL}$ and finite $X \subseteq \mathbf{A}$. Then each ω-embedding $\iota\colon \mathbf{A}|_X \hookrightarrow \mathbf{B}$ is an n-$\wedge\omega$-embedding for each $n \geq 1$.*

Proof. This is an immediate consequence of the fact that $a_1 \cdot b_1^{\omega} \wedge \cdots \wedge a_n \cdot b_n^{\omega} \leq^{\mathbf{A}} c$ if and only if $a_i \cdot b_i^{\omega} \leq^{\mathbf{A}} c$ for some $i \in \{1, \ldots, n\}$ for each totally ordered $\mathbf{A} \in \mathsf{ICRL}$. □

Let $\mathcal{P}_{\omega}(A)$ denote the set of finite subsets of A, and let $\mathbb{N}^{+} := \mathbb{N} - \{0\}$.

Lemma 7.14 (From $\wedge\pi$-embeddings to $\wedge\omega$-embeddings)**.**
Consider $\mathbf{A} \in \mathsf{FdICRL}$ and $\mathsf{K} \subseteq \mathsf{FdICRL}$. There is a function $f\colon \mathcal{P}_{\omega}(A) \to \mathbb{N}^{+}$ such that each finite partial subalgebra of $\mathbf{A}$ has a $\wedge\pi$-embedding into K if and only if each finite partial subalgebra $\mathbf{A}|_X$ has an $f(X)$-$\wedge\omega$-embedding into K.

Proof. The left-to-right direction holds for any f, since each $\wedge\pi$-embedding is an n-$\wedge\omega$-embedding for each $n \in \mathbb{N}$.

Conversely, we show that for each finite $X \subseteq \mathbf{A}$ we can find some finite extension $Y \subseteq \mathbf{A}$ of X and some $n \in \mathbb{N}$ such that each n-$\wedge\omega$-embedding of $\mathbf{A}|_Y$ into an algebra $\mathbf{B} \in \mathsf{K}$ restricts to a $\wedge\pi$-embedding of $\mathbf{A}|_X$ into $\mathbf{B}$. It will suffice to find for each $a \in X$ some finite extension $Y_a \subseteq \mathbf{A}$ of X and some $n_a \in \mathbb{N}$ such that for each $W \in \mathrm{Multi}_{\vee} X$ and each n_a-$\wedge\omega$-embedding ι of $\mathbf{A}|_{Y_a}$ into $\mathbf{B}$

$$(*) \qquad W^{\mathbf{A}} \leq^{\mathbf{A}} a \iff \iota(W)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a).$$

Taking $Y := \bigcup_{a \in X} Y_a$ and $n := \max_{a \in X} n_a$ then proves the lemma.

Given finite $X \subseteq \mathbf{A}$ and $a \in X$, let

$$U_a := \{W \in \mathrm{Multi}_{\vee} X \mid W^{\mathbf{A}} \leq^{\mathbf{A}} a\}.$$

Because $\mathbf{A}$ is integral, U_a is an upset of $\mathrm{Multi}_{\vee} X$. Higman's Lemma (Lemma 7.11) applied to U_a now provides $W_1, \ldots, W_l \in \mathrm{Multi}\, X$ such that for all $W \in \mathrm{Multi}_{\vee} X$

$$W \in U_a \iff W_1 \subseteq W \text{ or } \ldots \text{ or } W_l \subseteq W.$$

On the other hand, Lemma 7.12 applied to the downset $\mathrm{Multi}_{\vee} X - U_a$ provides $u_1, v_1, \ldots, u_k, v_k \in \mathrm{Multi}\, X$ such that for all $w \in \mathrm{Multi}\, X$

$$W \notin U_a \iff W \subseteq {\downarrow}(u_1 \oplus m v_1) \cup \cdots \cup {\downarrow}(u_k \oplus m v_k) \text{ for some } m \in \mathbb{N}.$$

In particular, $W_i^{\mathbf{A}} \leq^{\mathbf{A}} a$ for each $i \in \{1, \ldots, l\}$, while

$$u_1^{\mathbf{A}} \cdot (v_1^{\mathbf{A}})^{\omega} \wedge \cdots \wedge u_k \cdot (v_k^{\mathbf{A}})^{\omega} \nleq^{\mathbf{A}} a.$$

Each W_i is a downset of $\mathrm{Multi}\, X$ generated by the finite set $\max W_i$. We take

$$n_a := \max(k, |\max W_1|, \ldots, |\max W_l|)$$

and we take Y_a to be the extension of X by $w^{\mathbf{A}}$ where w ranges over all subwords (or at least over all initial segments and all letters) of $u_1, v_1, \ldots, u_k, v_k, w_1, \ldots, w_l$. This ensures that $\iota(u_1^{\mathbf{A}}) = \iota(u_1)^{\mathbf{B}}, \ldots, \iota(w_l^{\mathbf{A}}) = \iota(w_l)^{\mathbf{B}}$.

We now prove the left-to-right implication in $(*)$. Suppose that $W^{\mathbf{A}} \leq^{\mathbf{A}} a$ for $W \in \mathrm{Multi}_{\vee} X$. Then $W \in U_a$, so $W_i \subseteq W$ for some $i \in \{1, \ldots, l\}$, hence $\iota(W_i) \subseteq \iota(W)$ and by integrality $\iota(W)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(W_i)^{\mathbf{B}}$. Because ι is an n_a-$\wedge\omega$-embedding and $|\max W_i| \leq n_a$, the inequality $W_i^{\mathbf{A}} \leq^{\mathbf{A}} a$ implies that $\iota(W_i)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a)$, so $\iota(W)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(W_i)^{\mathbf{B}} \leq^{\mathbf{B}} \iota(a)$.

It remains to prove the right-to-left implication in $(*)$. Suppose that $W^{\mathbf{A}} \nleq^{\mathbf{A}} a$. Then $W \notin U$, so there is some $m \in \mathbb{N}$ such that

$$W \subseteq \downarrow(u_1 \oplus m v_1) \cup \cdots \cup \downarrow(u_k \oplus m v_k),$$

hence also

$$\iota(W) \subseteq \downarrow(\iota(u_1) \oplus m\iota(v_1)) \cup \cdots \cup \downarrow(\iota(u_k) \oplus m\iota(v_k)).$$

Consequently,

$$(u_1^{\mathbf{A}} \cdot (v_1^{\mathbf{A}})^m) \wedge \cdots \wedge (u_k^{\mathbf{A}} \cdot (v_k^{\mathbf{A}})^m) \leq^{\mathbf{A}} W^{\mathbf{A}}$$

and

$$(\iota(u_1)^{\mathbf{B}} \cdot \iota(v_1^{\mathbf{B}})^m) \wedge \cdots \wedge (\iota(u_k)^{\mathbf{B}} \cdot \iota(v_k^{\mathbf{B}})^m) \leq^{\mathbf{B}} \iota(W)^{\mathbf{B}}.$$

On the other hand, we know that

$$(u_1^{\mathbf{A}} \cdot (v_1^{\mathbf{A}})^{\omega}) \wedge \cdots \wedge (u_k^{\mathbf{A}} \cdot (v_k^{\mathbf{A}})^{\omega}) \nleq^{\mathbf{A}} a.$$

Because ι is an n_a-$\wedge\omega$-embedding and $k \leq n_a$,

$$(\iota(u_1)^{\mathbf{B}} \cdot (\iota(v_1)^{\mathbf{B}})^{\omega}) \wedge \cdots \wedge (\iota(u_k)^{\mathbf{B}} \cdot (\iota(v_k)^{\mathbf{B}})^{\omega}) \nleq^{\mathbf{B}} \iota(a).$$

The element $(\iota(u_1)^{\mathbf{B}} \cdot (\iota(v_1)^{\mathbf{B}})^{\omega}) \wedge \cdots \wedge (\iota(u_k)^{\mathbf{B}} \cdot (\iota(v_k)^{\mathbf{B}})^{\omega})$ therefore witnesses that indeed $\iota(W)^{\mathbf{B}} \nleq^{\mathbf{B}} \iota(a)$. □

Lemma 7.15. *Let $(\mathbf{A}_i)_{i \in I}$ be a finite family of* FdICRL*s, let $\mathbf{A} := \prod_{i \in I} \mathbf{A}_i$, and let $\pi_i \colon \mathbf{A} \to \mathbf{A}_i$ be the projection maps. Then for all $a_1, b_1, \ldots, a_n, b_n, c \in \mathbf{A}$*

$$a_1 \cdot b_1^{\omega} \wedge \cdots \wedge a_n \cdot b_n^{\omega} \nleq^{\mathbf{A}} c$$
$$\iff$$
$$\pi_i(a_1) \cdot \pi_i(b_1)^{\omega} \wedge \cdots \wedge \pi_i(a_n) \cdot \pi_i(b_n)^{\omega} \nleq^{\mathbf{A}_i} \pi_i(c) \text{ for some } i \in I.$$

Lemma 7.16 (Reduction to finite subdirect products)**.**
$\mathbb{C}\mathsf{x}_{\wedge}\mathbb{ISP}(\mathsf{K}) \subseteq \mathbb{UC}\mathsf{x}_{\wedge}\mathbb{SP}_{\mathrm{fin}}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$ *for each class* $\mathsf{K} \subseteq \mathsf{FdICRL}$.

Proof. The proof of Lemma 4.8 carries over. The only change is that the algebras $\mathbf{C}_{a,b,c}$ now need to be indexed by $(2n+1)$-tuples $a_1, b_1, \ldots, a_n, b_n, c \in X$ for $n := f(X)$, where the function $f \colon \mathcal{P}_{\omega}(A) \to \mathbb{N}^+$ comes from Lemma 7.14. □

Theorem 7.17 (Generating class for $\wedge$-conuclear V-algebras)**.**
Consider a variety $\mathsf{V} \subseteq \mathsf{FdICRL}$. The variety of $\wedge$-conuclear V-algebras is generated as a universal class by the $\wedge$-conuclear expansions of $\mathbb{ISP}_{\mathrm{fin}}(\mathsf{V}_{\mathrm{fsi}})$:

$$\mathbb{C}\mathsf{x}_{\wedge}(\mathsf{V}) = \mathbb{UC}\mathsf{x}_{\wedge}\mathbb{SP}_{\mathrm{fin}}(\mathsf{V}_{\mathrm{fsi}}).$$

Lemma 7.18. *Suppose that each finite partial subalgebra of each $\mathbf{A} \in \mathsf{K} \subseteq \mathsf{FdICRL}$ has a $\wedge\omega$-embedding into $\mathsf{L} \subseteq \mathsf{FdICRL}$. Then $\mathbb{C}\mathsf{x}_{\wedge}\mathbb{SP}_{\mathrm{fin}}(\mathsf{K}) \subseteq \mathbb{UC}\mathsf{x}_{\wedge}\mathbb{SP}_{\mathrm{fin}}(\mathsf{L})$.*

Theorem 7.19 (Generating class for $\wedge$-conuclear V-algebras)**.**
Consider a variety $\mathsf{V} \subseteq \mathsf{FdICRL}$. Suppose that each finite partial subalgebra of each $\mathbf{A} \in \mathsf{V}_{\mathrm{fsi}}$ has a $\wedge\omega$-embedding into $\mathsf{K} \subseteq \mathsf{FdICRL}$. Then the variety of $\wedge$-conuclear V-algebras is generated as a universal class by $\wedge$-conuclear $\mathbb{SP}_{\mathrm{fin}}(\mathsf{K})$-algebras:

$$\mathbb{C}\mathsf{x}_{\wedge}(\mathsf{V}) = \mathbb{UC}\mathsf{x}_{\wedge}\mathbb{SP}_{\mathrm{fin}}(\mathsf{K}).$$

Theorem 7.20 (Generation for Abelian ℓ-group cones with a $\wedge$-conucleus)**.**
The variety $\mathbb{C}\mathsf{x}(\mathsf{AbLG}^-)$ of Abelian ℓ-group cones with a $\wedge$-conucleus is generated as:

$$\mathbb{C}\mathsf{x}_\wedge(\mathsf{AbLG}^-) = \mathbb{U}\mathbb{C}\mathsf{x}_\wedge\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-).$$

Lemma 7.21. *For $\mathbf{A} := \mathrm{Lex}^-(n,\mathbb{Z})^k$ there are decision procedures (uniform in k and n) for the conditions $\langle\!\langle S \rangle\!\rangle^{\mathbf{A}} \cap {\downarrow} a = \emptyset$ for $a \in \mathbf{A}$ and $\langle\!\langle S \rangle\!\rangle^{\mathbf{A}} \cap ({\downarrow} a - {\downarrow} b)$ for $b \leq a$ in $\mathbf{A}$.*

Proof. The proof is entirely analogous to the proof of Lemma 5.11. The only further observation that we require is that it is decidable whether, given $a, b \in \mathbf{A}$ and $x_1, \dots, x_m \in X$, some finite meet of elements of the form $\alpha_1 x_1 + \dots + \alpha_n x_m$ for $\alpha_1, \dots, \alpha_m \in \mathbb{N}$ lies in ${\downarrow} a - {\downarrow} b$. But this holds because each condition of the form

$$\bigwedge_{1 \leq j \leq n} (\alpha_{j,1} x_{j,1} + \dots + \alpha_{j,m} x_{j,m}) \leq a$$

is equivalent to a Boolean combination of conditions of the forms

$$\alpha_{j,1} x_{j,1,i} + \dots + \alpha_{j,m} x_{j,m,i} < k,$$
$$\alpha_{j,1} x_{j,1,i} + \dots + \alpha_{j,m} x_{j,m,i} > k,$$

where $x_{j,k,i} := x_{j,k}(i)$. We already saw in the proof of Lemma 5.11 that it is decidable whether such coefficients α exist for any finite set of inequalities of these forms. □

Theorem 7.22 (Decidability for Abelian ℓ-group cones with a $\wedge$-conucleus)**.**
The universal theory of Abelian ℓ-group cones with a $\wedge$-conucleus is decidable.

Theorem 7.23 (Generation for Abelian ℓ-groups with a negative $\wedge$-conucleus)**.**
The variety of Abelian ℓ-groups with a negative $\wedge$-conucleus is generated as:

$$\mathbb{C}\mathsf{x}^-_\wedge(\mathsf{AbLG}) = \mathbb{U}\mathbb{C}\mathsf{x}^-_\wedge\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}).$$

Theorem 7.24 (Decidability for Abelian ℓ-groups with a negative $\wedge$-conucleus)**.**
The universal theory of Abelian ℓ-groups with a negative $\wedge$-conucleus is decidable.

Theorem 7.25 (The Montagna–Tsinakis representation [37])**.**
The $\wedge$-conuclear images of Abelian ℓ-groups (Abelian ℓ-group cones) are precisely the fully distributive (integral) cancellative commutative residuated lattices.

Theorem 7.26 (Generation for fully distributive cancellative ICRLs)**.**
The variety $\mathbb{C}_\wedge(\mathsf{AbLG}^-)$ of fully distributive cancellative ICRLs is generated as:

$$\mathbb{C}_\wedge(\mathsf{AbLG}^-) = \mathbb{U}\mathbb{C}_\wedge\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-).$$

Theorem 7.27 (Decidability for fully distributive cancellative ICRLs)**.**
The universal theory of fully distributive cancellative ICRLs is decidable.

We now move on to the results of Section 6. To reproduce these results for $\wedge$-conuclei, it suffices to make two observations. Firstly, given a $\wedge$-conucleus $\Box$ and a nucleus $\Diamond$ on the conuclear residuated lattice $\langle \mathbf{A}, \Box \rangle$, the restriction of $\Box$ to $\mathbf{A}_\Diamond$ is a $\wedge$-conucleus. Secondly, given a meet preserving nucleus $\Diamond$ on $\mathbf{A}$ and a $\wedge$-conucleus $\Box$ on $\mathbf{A}_\Diamond$, the conucleus $\overline{\Box} x := \Box \Diamond x \wedge x$ is a $\wedge$-conucleus.

Theorem 7.28. *For each class K of distributive integral residuated lattices*

$$\mathbb{I}\mathrm{nt}(\mathbb{C}\mathsf{x}_\wedge(\mathsf{K})) = \mathbb{C}\mathsf{x}_\wedge(\mathbb{I}\mathrm{nt}(\mathsf{K})), \qquad \mathbb{I}\mathrm{nt}(\mathbb{C}_\wedge(\mathsf{K})) = \mathbb{C}_\wedge(\mathbb{I}\mathrm{nt}(\mathsf{K})).$$

Theorem 7.29 (The Young representation)**.**
$\wedge$-conuclear MV-algebras are precisely the unit intervals of $\wedge$-conuclear Abelian ℓ-group cones. Consequently, the $\wedge$-conuclear images of MV-algebras are precisely the unit intervals of fully distributive cancellative ICRL*s.*

Theorem 7.30 (Generation for $\wedge$-conuclear MV-algebras)**.**
The variety of $\wedge$-conuclear MV-algebras is generated as a universal class as follows:

$$\mathbb{C}\mathrm{x}_\wedge(\mathsf{MV}) = \mathbb{U}\mathbb{C}\mathrm{x}_\wedge\mathbb{P}_{\mathrm{fin}}(\mathbb{I}\mathrm{nt}(\mathbb{Z}^-)) = \mathbb{U}\mathbb{C}\mathrm{x}_\wedge(\mathsf{MV}_{\mathrm{fin}}).$$

In particular, it has the Finite Embeddability Property.

Theorem 7.31 (Decidability for $\wedge$-conuclear MV-algebras)**.**
The universal theory of $\wedge$-conuclear MV-algebras is decidable.

It remains an open problem to describe the $\wedge$-conuclear images of MV-algebras.

8. Join preserving conuclei

In this final section, we consider join preserving conuclei and, as a related matter, conuclei on totally ordered algebras. The results stated in this section will be straightforward consequences of results proved in the previous sections. We in particular recover the main result of Horčík [29].

Recall that a *join preserving conucleus* or a *$\vee$-conucleus* for short is a conucleus which satisfies the equation $\Box(x \vee y) = \Box x \vee \Box y$. The class operators $\mathbb{C}_\vee$, $\mathbb{C}\mathrm{x}_\vee$, and $\mathbb{C}\mathrm{x}_\vee^-$ are defined as expected.

Lemma 8.1. *Let $\langle \mathbf{A}, \Box \rangle$ be a $\vee$-conuclear* CRL*. Then $\langle \mathbf{A}, \Box \rangle$ is finitely subdirectly irreducible if and only if $\mathbf{A}$ is finitely subdirectly irreducible.*

Proof. This immediately follows from Lemmas 2.5 and 2.9, since $\Box x \vee \Box y = 1$ for $x, y \leq 1$ in $\mathbf{A}$ is equivalent to $\Box(x \vee y) = 1$ and therefore to $x \vee y = 1$. ☐

Recall that SemCRL denotes the variety of semilinear CRLs.

Lemma 8.2. $\mathbb{C}\mathrm{x}_\vee(\mathsf{K}) = \mathbb{I}\mathbb{S}\mathbb{P}(\mathbb{C}\mathrm{x}_\vee(\mathsf{K}_{\mathrm{fsi}}))$ *for each variety* $\mathsf{K} \subseteq \mathsf{CRL}$*. If* $\mathsf{K} \subseteq \mathsf{SemCRL}$*, then* $\mathbb{C}\mathrm{x}_\vee(\mathsf{K})$ *is the variety* $\mathbb{I}\mathbb{S}\mathbb{P}(\mathbb{C}\mathrm{x}(\mathsf{K}_{\mathrm{fsi}}))$ *of semilinear conuclear* K*-algebras.*

Proof. The first claim follows immediately from Lemma 8.1, since $\mathbb{C}\mathrm{x}_\vee(\mathsf{K})$ is a variety and therefore $\mathbb{C}\mathrm{x}_\vee(\mathsf{K}) = \mathbb{I}\mathbb{S}\mathbb{P}((\mathbb{C}\mathrm{x}_\vee(\mathsf{K}))_{\mathrm{fsi}}) = \mathbb{I}\mathbb{S}\mathbb{P}(\mathbb{C}\mathrm{x}_\vee(\mathsf{K}_{\mathrm{fsi}}))$. If $\mathsf{K} \subseteq \mathsf{SemCRL}$, then $\mathbb{C}\mathrm{x}_\vee(\mathsf{K}_{\mathrm{fsi}}) = \mathbb{C}\mathrm{x}(\mathsf{K}_{\mathrm{fsi}})$, since each conucleus on a totally ordered residuated lattice is a $\vee$-conucleus. ☐

In the following, we use the notation

$$\mathsf{Lex} := \{\mathrm{Lex}(n, \mathbb{Z}) \mid n \in \mathbb{N}\}.$$

Theorem 8.3 (Generation and decidability for totally ordered algebras)**.**
The universal class of

(i) totally ordered conuclear Abelian ℓ-group cones is generated as:

$$\mathbb{C}\mathrm{x}(\mathsf{AbLG}^-_{\mathrm{fsi}}) = \mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathsf{Lex}^-).$$

(ii) totally ordered Abelian ℓ-groups with a negative conucleus is generated as:

$$\mathbb{C}\mathrm{x}(\mathsf{AbLG}_{\mathrm{fsi}}) = \mathbb{U}\mathbb{C}\mathrm{x}^-\mathbb{S}(\mathsf{Lex}).$$

(iii) totally ordered conuclear MV-algebras is generated as:

$$\mathbb{C}\mathrm{x}(\mathsf{MV}_{\mathrm{fsi}}) = \mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathbb{I}\mathrm{nt}(\mathsf{Lex}^-)).$$

(iv) totally ordered cancellative ICRL*s is generated as:*

$$\mathbb{C}(\mathsf{AbLG}^-_{\mathrm{fsi}}) = \mathbb{U}\mathbb{C}\mathbb{S}(\mathsf{Lex}^-).$$

All of these universal classes have a decidable universal theory.

Proof. The right-to-left inclusions hold because the left-hand sides are universal classes. It remains to prove the left-to-right inclusions.

(i): each finite partial subalgebra of each algebra in $\mathsf{AbLG}^-_{\mathrm{fsi}}$ has a weak ω-embedding into Lex^- by Lemma 5.7, so the inclusion $\mathbb{C}\mathsf{x}(\mathsf{AbLG}^-_{\mathrm{fsi}}) \subseteq \mathbb{U}\mathbb{C}\mathsf{x}\mathbb{S}(\mathsf{Lex}^-)$ follows by Lemmas 4.3 and 4.4 together with Lemma 3.7.

(ii): this follows immediately from (i) and Lemma 5.12.

(iii): $\mathbb{C}\mathsf{x}(\mathsf{MV}_{\mathrm{fsi}}) = \mathbb{C}\mathsf{x}(\mathbb{I}\mathrm{nt}(\mathsf{AbLG}^-_{\mathrm{fsi}})) = \mathbb{I}\mathrm{nt}(\mathbb{C}\mathsf{x}(\mathsf{AbLG}^-_{\mathrm{fsi}})) \subseteq \mathbb{I}\mathrm{nt}(\mathbb{U}\mathbb{C}\mathsf{x}\mathbb{S}(\mathsf{Lex}^-)) \subseteq \mathbb{U}\mathbb{C}\mathsf{x}\mathbb{S}(\mathbb{I}\mathrm{nt}(\mathsf{Lex}^-))$, using (i) and Lemma 6.2 and Theorem 6.5.

(iv): let $\mathbb{I}\mathrm{mg}$ be the class operator which maps a class of conuclear residuated lattices $\langle \mathbf{A}, \Box \rangle$ to the class of their conuclear images $\mathbf{A}_\Box$. Clearly $\mathbb{C} = \mathbb{I}\mathrm{mg}\,\mathbb{C}\mathsf{x}$. Then $\mathbb{C}(\mathsf{AbLG}^-_{\mathrm{fsi}}) = \mathbb{I}\mathrm{mg}\,\mathbb{C}\mathsf{x}(\mathsf{AbLG}^-_{\mathrm{fsi}}) \subseteq \mathbb{I}\mathrm{mg}\,\mathbb{U}\mathbb{C}\mathsf{x}\mathbb{S}(\mathsf{Lex}^-) \subseteq \mathbb{U}\,\mathbb{I}\mathrm{mg}\,\mathbb{C}\mathsf{x}\mathbb{S}(\mathsf{Lex}^-) = \mathbb{U}\mathbb{C}\mathbb{S}(\mathsf{Lex}^-)$ using (i), since $\mathbb{I}\mathrm{mg}$ commutes with $\mathbb{I}$ and with $\mathbb{P}_\mathrm{U}$ and $\mathbb{I}\mathrm{mg}\,\mathbb{S}(\mathsf{K}) \subseteq \mathbb{S}\,\mathbb{I}\mathrm{mg}(\mathsf{K})$ for each class K of conuclear residuated lattices.

The decidability of (i) is proved by replacing $(\mathbb{Z}^-)^k$ by $\mathrm{Lex}^-(n, \mathbb{Z})$ in the proof of Theorem 5.10 (more precisely, in the variant which uses Lemma 5.11). The decidability of (ii) follows from the decidability of (i) by Lemma 5.12. The decidability of (iii) follows from the decidability of (i) by the totally ordered case of Theorem 6.7 and Lemma 6.9. Finally, the decidability of (iv) follows from the decidability of (i) by Lemma 2.7. □

Recall that $\mathbb{Q}(\mathsf{K}) := \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_\mathrm{U}(\mathsf{K})$ is the quasivariety generated by the class K.

Theorem 8.4 (Generation and decidability for semilinear algebras)**.**
The variety of

(i) $\vee$-conuclear Abelian ℓ-group cones is generated as:

$$\mathbb{C}\mathsf{x}_\vee(\mathsf{AbLG}^-) = \mathbb{Q}\mathbb{C}\mathsf{x}\mathbb{S}(\mathsf{Lex}^-).$$

(ii) Abelian ℓ-groups with a negative $\vee$-conucleus is generated as:

$$\mathbb{C}\mathsf{x}^-_\vee(\mathsf{AbLG}) = \mathbb{Q}\mathbb{C}\mathsf{x}^-\mathbb{S}(\mathsf{Lex}).$$

(iii) $\vee$-conuclear MV-algebras is generated as:

$$\mathbb{C}\mathsf{x}_\vee(\mathsf{MV}) = \mathbb{Q}\mathbb{C}\mathsf{x}\mathbb{S}(\mathbb{I}\mathrm{nt}(\mathsf{Lex}^-)).$$

(iv) semilinear cancellative ICRL*s is generated as:*

$$\mathbb{C}_\vee(\mathsf{AbLG}^-) = \mathbb{Q}\mathbb{C}\mathbb{S}(\mathsf{Lex}^-).$$

All of these varieties have a decidable universal theory.

Proof. The right-to-left inclusions hold because the left-hand sides are varieties. It remains to prove the left-to-right inclusions.

(i): Lemma 8.2 and Theorem 8.3(i) imply that $\mathbb{C}\mathsf{x}_\vee(\mathsf{AbLG}^-) \subseteq \mathbb{Q}\mathbb{C}\mathsf{x}(\mathsf{AbLG}^-_{\mathrm{fsi}}) \subseteq \mathbb{Q}\mathbb{U}\mathbb{C}\mathsf{x}\mathbb{S}(\mathsf{Lex}^-) = \mathbb{Q}\mathbb{C}\mathsf{x}\mathbb{S}(\mathsf{Lex}^-)$.

(ii): this follows immediately from (i) and Lemma 5.12.

(iii): Lemma 8.2 and Theorem 8.3(iii) imply that $\mathbb{C}\mathsf{x}_\vee(\mathsf{MV}) \subseteq \mathbb{Q}\mathbb{C}\mathsf{x}(\mathsf{MV}_{\mathrm{fsi}}) \subseteq \mathbb{Q}\mathbb{U}\mathbb{C}\mathsf{x}\mathbb{S}(\mathbb{I}\mathrm{nt}(\mathsf{Lex}^-)) = \mathbb{Q}\mathbb{C}\mathsf{x}\mathbb{S}(\mathbb{I}\mathrm{nt}(\mathsf{Lex}^-))$.

(iv): this follows immediately from Theorem 8.3(iv).

The decidability claim follows from Theorem 8.3 and Lemma 2.1. □

The generation and decidability result for semilinear cancellative ICRLs stated in the above corollary is precisely the main decidability result of Horčík [29].

The question naturally arises whether considering conuclear expansions of non-Archimedean totally ordered Abelian ℓ-groups is necessary, i.e. whether one may replace $\mathrm{Lex}(n, \mathbb{Z})$ by $\mathbb{Z}$, or at least by $\mathbb{R}$. The following result shows that this is not possible: the above generation results for totally ordered and semilinear algebras require lexicographic powers of unbounded depth.

Theorem 8.5. *For each $n \in \mathbb{N}$:*

(i) $\mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R})) \subsetneq \mathbb{C}\mathrm{x}(\mathsf{AbLG}^{-}_{\mathrm{fsi}})$.
(ii) $\mathbb{U}\mathbb{C}\mathrm{x}^{-}\mathbb{S}(\mathrm{Lex}(n, \mathbb{R})) \subsetneq \mathbb{C}\mathrm{x}^{-}(\mathsf{AbLG}_{\mathrm{fsi}})$.
(iii) $\mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathbb{I}\mathrm{nt}(\mathrm{Lex}^{-}(n, \mathbb{R}))) \subsetneq \mathbb{C}\mathrm{x}(\mathsf{MV}_{\mathrm{fsi}})$.
(iv) $\mathbb{U}\mathbb{C}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R})) \subsetneq \mathbb{C}(\mathsf{AbLG}^{-}_{\mathrm{fsi}})$.

Consequently, for each $n \in \mathbb{N}$:

(v) $\mathbb{Q}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R})) \subsetneq \mathbb{C}\mathrm{x}_{\vee}(\mathsf{AbLG}^{-})$.
(vi) $\mathbb{Q}\mathbb{C}\mathrm{x}^{-}\mathbb{S}(\mathrm{Lex}(n, \mathbb{R})) \subsetneq \mathbb{C}\mathrm{x}^{-}_{\vee}(\mathsf{AbLG})$.
(vii) $\mathbb{Q}\mathbb{C}\mathrm{x}\mathbb{S}(\mathbb{I}\mathrm{nt}(\mathrm{Lex}^{-}(n, \mathbb{R}))) \subsetneq \mathbb{C}\mathrm{x}_{\vee}(\mathsf{MV})$.
(viii) $\mathbb{Q}\mathbb{C}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R})) \subsetneq \mathbb{C}_{\vee}(\mathsf{AbLG}^{-})$.

Proof. Claims (v)–(viii) follow from claims (i)–(iv) by the analogue of Jónsson's Lemma for quasivarieties [20, Lemma 1.5], which states that for each class of algebras K the relatively finitely subdirectly irreducible algebras in $\mathbb{Q}(\mathsf{K})$ in fact lies in $\mathbf{A} \in \mathbb{U}(\mathsf{K})$. For example, the inclusion $\mathbb{Q}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R})) \subseteq \mathbb{C}\mathrm{x}_{\vee}(\mathsf{AbLG}^{-})$ is equivalent to the claim that each relatively finitely subdirectly irreducible algebra of $\mathbb{Q}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R}))$ is in $\mathbb{C}\mathrm{x}_{\vee}(\mathsf{AbLG}^{-})$, since $\mathbb{C}\mathrm{x}_{\vee}(\mathsf{AbLG}^{-})$ is a quasivariety. But each such algebra lies in $\mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R}))$ by [20, Lemma 1.5], and conversely each algebra in $\mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R}))$ is totally ordered and therefore finitely subdirectly irreducible by Lemmas 2.5 and 8.1. The inclusion $\mathbb{Q}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R})) \subseteq \mathbb{C}\mathrm{x}_{\vee}(\mathsf{AbLG}^{-})$ is thus equivalent to $\mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R})) \subseteq \mathbb{C}\mathrm{x}_{\vee}(\mathsf{AbLG}^{-})$. This is in turn equivalent to $\mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R})) \subseteq \mathbb{C}\mathrm{x}_{\vee}(\mathsf{AbLG}^{-}_{\mathrm{fsi}})$, since $\mathsf{AbLG}^{-}_{\mathrm{fsi}}$ is the class of totally ordered algebras in AbLG^{-}.

Claim (i) follows from (iv), since a universal sentence valid in $\mathbb{U}\mathbb{C}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R}))$ but not in $\mathbb{C}(\mathsf{AbLG}^{-}_{\mathrm{fsi}})$ yields a universal sentence valid in $\mathbb{U}\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R}))$ but not in $\mathbb{C}\mathrm{x}(\mathsf{AbLG}^{-}_{\mathrm{fsi}})$ by Lemma 2.7. On the other hand, the proof of (iv) is best understood as a slight technical modification of the proof of (i). We therefore find it instructive to start with a direct proof of (i).

(i): we show that there is a universal sentence ψ_n satisfied in $\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n, \mathbb{R}))$ but not in $\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^{-}(n+1), \mathbb{Z})$. We define this sentence as follows:

$$\begin{aligned} \phi(x, y) &:= (y \leq \Box x) \ \& \ (\Box y < y + \Box x), \\ \phi_n(x_0, \ldots, x_{n+1}) &:= \phi(x_0, x_1) \ \& \ \phi(x_1, x_2) \ \& \ \ldots \ \& \ \phi(x_n, x_{n+1}), \\ \psi_n &:= \forall x_0, \ldots, x_{n+1} \, \neg \phi_n(x_0, \ldots, x_{n+1}). \end{aligned}$$

We first show that if $\phi(a, b)$ holds in a totally ordered conuclear Abelian ℓ-group cone $\langle \mathbf{G}^{-}, \Box \rangle$, then $\omega(\Box a) \not\leq b$. Suppose for the sake of contradiction that $\phi(a, b)$ but $\omega(\Box a) \leq b$. Because $b \leq \Box a$, there is some $k \in \mathbb{N}$ such that $(k+1)(\Box a) < b \leq k(\Box a)$. Observe that $(k+1)(\Box a) \in \mathbf{G}^{-}_{\Box}$, so $(k+1)(\Box a) \leq \Box b$. But the inequality

$\Box b < b + \Box a$ now implies that $(k+1)(\Box a) \leq \Box b < b + \Box a \leq k(\Box a) + \Box a = (k+1)(\Box a)$, which is a contradiction.

Consequently, if there is no sequence $a_0, \dots, a_{n+1} \in \mathbf{G}^-_\Box$ such that $\omega(\Box a_i) \nleq a_{i+1}$ for $i \in \{1, \dots, n\}$, then $\langle \mathbf{G}^-, \Box \rangle$ satisfies ψ_n. But $\omega(\Box a_i) \nleq a_{i+1}$ implies $\omega a_i \nleq a_{i+1}$, and there is indeed no sequence $a_0, \dots, a_{n+1} \in \mathrm{Lex}^-(n, \mathbb{R})$ such that $\omega a_i \nleq a_{i+1}$. Thus each algebra in $\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^-(n, \mathbb{R}))$ satisfies ψ_n.

On the other hand, there is some sequence $a_0, \dots, a_{n+1} \in \mathrm{Lex}^-(n+1, \mathbb{Z})$ such that $\omega a_i \nleq a_{i+1}$ for $i \in \{1, \dots, n\}$. Necessarily $a_0 := 0$ and a_{n+1} is cofinal in $\mathrm{Lex}^-(n+1, \mathbb{Z})$. Let $S := \{2a_0, \dots, 2a_{n+1}\}$. Then the conuclear residuated lattice $\langle \mathrm{Lex}^-(n+1, \mathbb{Z}), \Box_S \rangle \in \mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^-(n+1, \mathbb{Z}))$ invalidates the universal sentence ψ_n under the valuation $x_i \mapsto a_i$, since $\phi(a_i, a_{i+1})$ for each $i \in \{0, \dots, n\}$.

(ii): this follows from (i), since each universal sentence valid in $\mathbb{C}\mathrm{x}\mathbb{S}(\mathrm{Lex}^-(n, \mathbb{R}))$ but not in $\mathbb{C}\mathrm{x}(\mathsf{AbLG}^-_{\mathrm{fsi}})$ yields a universal sentence valid in $\mathbb{C}\mathrm{x}^-\mathbb{S}(\mathrm{Lex}(n, \mathbb{R}))$ but not in $\mathbb{C}\mathrm{x}(\mathsf{AbLG}_{\mathrm{fsi}})$ by the analogue of Lemma 2.7 for negative cones of conuclear residuated lattices.

(iii): the proof of (i) also establishes (iii) if we replace $+$ by $\oplus$ and take the interval $[2a_{n+1}, 0]$ in the last paragraph of the proof.

(iv): we show that there is a universal sentence ψ_n satisfied in $\mathbb{C}\mathbb{S}(\mathrm{Lex}^-(n, \mathbb{R}))$ but not in $\mathbb{C}\mathbb{S}(\mathrm{Lex}^-(n+1), \mathbb{Z})$. We define this sentence as follows:

$$\phi(x, u, v) := (u \wedge v \leq u + x) \,\&\, ((v \ominus u) + u < (u \wedge v) + x),$$
$$\phi_n(u_0, v_0, \dots, u_{n+1}, v_{n+1}) := \phi(v_0 \ominus u_0, u_1, v_1) \,\&\, \dots \,\&\, \phi(v_n \ominus u_n, u_{n+1}, v_{n+1}),$$
$$\psi_n := \forall u_0, v_0, \dots, u_{n+1}, v_{n+1} \, \neg \phi_n(u_0, v_0, \dots, u_{n+1}, v_{n+1}).$$

By Theorem 5.16 each totally ordered cancellative ICRL has the form $\mathbf{A} := \mathbf{G}^-_\Box$, where $\mathbf{G}$ is a totally ordered Abelian ℓ-group and $\Box$ is a conucleus on $\mathbf{G}^-$. Observe that $\phi(c, a, b)$ holds in $\mathbf{A}$ if and only if

$$(b \ominus^{\mathbf{G}^-} a \leq c) \,\&\, (b \ominus^{\mathbf{A}} a < (b \ominus^{\mathbf{G}^-} a) + c),$$

since the inequality $b \wedge c \leq b + a$ is equivalent to $1 \wedge (c - b) \leq a$, and $b + (c \ominus^{\mathbf{A}} b) < (b \wedge c) + a$ is equivalent to $c \ominus^{\mathbf{A}} b < (1 \wedge (c - b)) + a$.

We first show that if $\phi(c, a, b)$ holds in $\mathbf{A}$, then $\omega c \nleq b \ominus^{\mathbf{A}} a$. Observe that $kc \leq b \ominus^{\mathbf{A}} a$ if and only if $kc \nleq b \ominus^{\mathbf{G}^-} a$, since $kc \in \mathbf{A}$. Suppose therefore for the sake of contradiction that $\phi(c, a, b)$ holds in $\mathbf{A}$ but $\omega c \leq b \ominus^{\mathbf{G}^-} a$. Because $b \ominus^{\mathbf{G}^-} a \leq c$ and $\omega c \leq b \ominus^{\mathbf{G}^-} a$, there is some $k \in \mathbb{N}$ such that $(k+1)c < b \ominus^{\mathbf{G}^-} a \leq kc$. But the inequality $b \ominus^{\mathbf{A}} a < (b \ominus^{\mathbf{G}^-} a) + c$ now implies that $(k+1)c < b \ominus^{\mathbf{A}} a < (b \ominus^{\mathbf{G}^-} a) + c \leq kc + c = (k+1)c$, which is a contradiction.

Consequently, if there is no sequence $a_0, b_0, \dots, a_{n+1}, b_{n+1} \in \mathbf{A}$ such that $\omega c_i \nleq c_{i+1}$ for $i \in \{0, \dots, n\}$ for $c_i := b_i \ominus a_i$, then $\mathbf{A}$ satisfies ψ_n. But there is in fact no sequence $c_0, \dots, c_{n+1} \in \mathrm{Lex}^-(n, \mathbb{R})$ whatsoever such that $\omega c_i \nleq c_{i+1}$. Thus each algebra in $\mathbb{C}\mathbb{S}(\mathrm{Lex}^-(n, \mathbb{R}))$ satisfies ψ_n.

On the other hand, there is some sequence $c_0, \dots, c_{n+1} \in \mathbf{G}^- := \mathrm{Lex}^-(n+1, \mathbb{Z})$ such that $\omega c_i \nleq c_{i+1}$ for $i \in \{0, \dots, n\}$. Necessarily $c_0 := 0$ and c_{n+1} is cofinal in $\mathbf{G}^-$. Take $a_i, b_i \leq 2c_i$ such that $b_i \ominus^{\mathbf{G}^-} a_i = c_i$, with $a_0 := 0$ and $b_0 := 0$. Let $S := \{2c_0, \dots, 2c_{n+1}, a_0, b_0, \dots, a_{n+1}, b_{n+1}\}$. Take $\mathbf{A} := (\mathbf{G}^-)_{\Box_S}$. Then $b_i \ominus^{\mathbf{A}} a_i = c_i$, so $b_{i+1} \ominus^{\mathbf{G}^-} a_{i+1} \leq c_i$ and $b_{i+1} \ominus^{\mathbf{A}} a_{i+1} < (b_{i+1} \ominus^{\mathbf{G}^-} a_{i+1}) + c_i$. As we have seen, these inequalities together are equivalent to $\phi(c_i, a_{i+1}, b_{i+1})$ being true in $\mathbf{A}$. Thus

$\mathbf{A} \in \mathbb{CS}(\mathrm{Lex}^-(n+1, \mathbb{Z}))$ invalidates the universal sentence ψ_n under the valuation $x_i \mapsto b_i \ominus^{\mathbf{A}} a_i$ and $u_i \mapsto a_i$ and $v_i \mapsto b_i$. □

Corollary 8.6. *The variety* $\mathbb{C}\mathsf{x}_{\vee}(\mathsf{MV})$ *does not have the FEP.*

Proof. Each finite algebra in $\mathbb{C}\mathsf{x}_{\vee}(\mathsf{MV})$ embeds into a product of totally ordered finite algebras in $\mathbb{C}\mathsf{x}_{\vee}(\mathsf{MV})$ by Lemma 8.2. Each such totally ordered algebra lies in $\mathbb{C}\mathsf{x}\mathbb{S}(\mathbb{I}\mathrm{nt}(\mathrm{Lex}^-(n, \mathbb{R})))$, indeed in $\mathbb{C}\mathsf{x}(\mathbb{I}\mathrm{nt}(\mathrm{Lex}^-(n, \mathbb{Z})))$, so each finite algebra in $\mathbb{C}\mathsf{x}_{\vee}(\mathsf{MV})$ lies $\mathbb{QC}\mathsf{x}\mathbb{S}(\mathrm{Lex}^-(1, \mathbb{R}))$. Consequently, $\mathbb{C}\mathsf{x}(\mathsf{MV})$ fails to have the FEP by Theorem 8.5. □

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References

[1] Kempton Albee and Nick Galatos. A recipe for canonical formulas in substructural logics. Unpublished manuscript.

[2] Shun'ichi J. Amano. The finite embeddability property of the variety of modal residuated lattices. In *Proceedings of the 39th MLG meeting at Gamagori, Japan 2005*, pages 39–42, 2005.

[3] Shun'ichi J. Amano. The finite embeddability property for some modal algebras. Master's thesis, Japan Advanced Institute of Science and Technology, School of Information Science, 2006. Available at `https://dspace.jaist.ac.jp/dspace/handle/10119/1982`.

[4] Marlow Anderson and Todd Feil. *Lattice-ordered groups: an introduction*. Number 4 in Reidel Texts in Mathematical Sciences. D. Reidel Publishing Company, 1988.

[5] Patrick Bahls, Jac Cole, Nikolaos Galatos, Peter Jipsen, and Constantine Tsinakis. Cancellative residuated lattices. *Algebra Universalis*, 50(1):83–106, 2003.

[6] Clifford Bergman. *Universal algebra: Fundamentals and selected topics*. CRC Press, 2011.

[7] Guram Bezhanishvili. Varieties of monadic Heyting algebras I. *Studia Logica*, 61(3):367–402, 1998.

[8] Guram Bezhanishvili. Locally finite varieties. *Algebra Universalis*, 46:531–548, 2001.

[9] Garrett Birkhoff. *Lattice theory*. Number XXV in Colloquium Publications. American Mathematical Society, 3rd edition, 1967.

[10] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal logic*. Cambridge University Press, 2001.

[11] Willem J. Blok and Clint J. van Alten. The finite embeddability property for residuated lattices, pocrims and BCK-algebras. *Algebra Universalis*, 48:253–271, 2002.

[12] Stanley Burris. Polynomial time uniform word problems. *Mathematical Logic Quarterly*, 41(2):173–182, 1995.

[13] Stanley Burris and Hanamantagouda P. Sankappanavar. *A Course in Universal Algebra*. Springer, 1981.

[14] Xavier Caicedo, George Metcalfe, Ricardo Rodríguez, and Jonas Rogger. A finite model property for Gödel modal logics. In Leonid Libkin, Ulrich Kohlenbach, and Ruy de Queiroz, editors, *Logic, Language, Information, and Computation. 20th International Workshop, WoLLIC 2013 Darmstadt, Germany, August 20-23, 2013 Proceedings*, number 8071 in Lecture Notes in Computer Science, pages 226–237. Springer, 2013.

[15] Riquelmi Cardona and Nikolaos Galatos. The finite embeddability property for noncommutative knotted extensions of RL. *International Journal of Algebra and Computation*, 25(3):349–379, 2015.

[16] Alexander Chagrov and Michael Zakharyaschev. *Modal logic*. Oxford University Press, 1997.

[17] Roberto L. O. Cignoli, Itala M. L. D'Ottaviano, and Daniele Mundici. *Algebraic foundations of many-valued reasoning*. Number 7 in Trends in Logic. Studia Logica Library. Springer, 2000.

[18] Petr Cintula, George Metcalfe, and Naomi Tokuda. One-variable fragments of first-order logics. *The Bulletin of Symbolic Logic*, 30(2):253–278, 2024.

[19] Stavros C. Cosmadakis. The word and generator problems for lattices. *Information and Computation*, 77(3):192–217, 1988.

[20] Janusz Czelakowski and Wiesław Dziobiak. Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class. *Algebra Universalis*, 27:128–149, 1990.

[21] Jim de Groot and Ian Shillito. Intuitionistic S4 as a logic of topological spaces. *Journal of Logic and Computation*, 35(7), 2025.

[22] Nick Galatos and Adam Přenosil. Complemented MacNeille completions and algebras of fractions. *Journal of Algebra*, 623:288–357, 2023.

[23] Nikolaos Galatos and Rostislav Horčík. Cayley's and Holland's theorems for idempotent semirings and their applications to residuated lattices. *Semigroup Forum*, 87(3):569–589, 2013.

[24] Nikolaos Galatos and Peter Jipsen. Residuated frames with applications to decidability. *Transactions of the American Mathematical Society*, 365(3):1219–1249, 2013.

[25] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated lattices: an algebraic glimpse at substructural logics*. Number 151 in Studies in logic and the foundations of mathematics. Elsevier, 2007.

[26] A. M. W. Glass and Yuri Gurevich. The word problem for lattice-order groups. *Transactions of the American Mathematical Society*, 280(1):127–138, 1983.

[27] Hans Hahn. Über die nichtarchimedischen Größensysteme. *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Mathematisch – Naturwissenschaftliche Klasse*, 116(IIa):601–655, 1907.

[28] Graham Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, 3(2):326–336, 1952.

[29] Rostislav Horčík. Decidability of cancellative extension of monoidal t-norm based logic. *Logic Journal of the IGPL*, 14(6):827–843, 2006.

[30] H. B. Hunt III, D. J. Rosenkrantz, and P. A. Bloniarz. On the computational complexity of algebra on lattices. *SIAM Journal on Computing*, 16(1):129–148, 1987.

[31] Richard E. Ladner. The computational complexity of provability in systems of modal logic. *SIAM Journal on Computing*, 6(3):467–480, 1977.

[32] Anatolii Ivanovich Mal'cev. *Algebraic systems*. Number 192 in Grundlehren der Mathematischen Wissenschaften. Springer, 1973.

[33] Ralph McKenzie. An algebraic version of categorical equivalence for varieties and more generalalgebraic categories. In Aldo Ursini and Paolo Aglianò, editors, *Logic and algebra*, number 180 in Lecture notes in pure and applied mathematics. Marcel Dekker, 1996.

[34] J. C. C. McKinsey. A solution of the decision problem for the Lewis systems S2 and S4. *The Journal of Symbolic Logic*, 6(4):117–134, 1941.

[35] J. C. C. McKinsey and Alfred Tarski. On closed elements in closure algebras. *Annals of Mathematics*, 47(1):122–162, 1946.

[36] George Metcalfe, Francesco Paoli, and Constantine Tsinakis. *Residuated structures in algebra and logic*. Number 277 in Mathematical Surveys and Monographs. American Mathematical Society, 2023.

[37] Franco Montagna and Constantine Tsinakis. Ordered groups with a conucleus. *Journal of Pure and Applied Algebra*, 214(1):71–88, 2010.

[38] Daniele Mundici. Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus. *Journal of Functional Analysis*, 65:15–63, 1986.

[39] Daniele Mundici. *Advanced Łukasiewicz calculus and MV-algebras*. Number 35 in Trends in Logic. Studia Logica Library. Springer, 2011.

[40] Jeffrey S. Olson. Subdirectly irreducible residuated semilattices and positive universal classes. *Studia Logica*, 83(1):393–406, 2006.

[41] Adam Přenosil. Pointed lattice subreducts of varieties of residuated lattices. *Order*, 42:87–114, 2025.

[42] Robert W. Quackenbush. Completeness theorems for universal and implicational logics of algebras via congruences. *Proceedings of the American Mathematical Society*, 103(4):1015–1021, 1988.

[43] Thoralf Skolem. Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit und Beweisbarkeit mathematischen Sätze nebst einem Theorem über dichte Mengen. *Videnskabsselskabets skrifter. I, Matematisk-naturvidenskabelig klasse*, 4:1–36, 1920.

[44] Hiroki Takamura. The variety of modal $\mathbf{FL}_{ew}$-algebras is generated by its finite simple members. In Guide Governatori, Ian Hodkinson, and Yde Venema, editors, *Advances in Modal Logic, Volume 6*, pages 469–479. College Publications, 2006.

[45] Hiroki Takamura. Semisimplicity, EDPC and discriminator varieties of bounded weak-commutative residuated lattices with an S4-like modal operator. *Studia Logica*, 100:1137–1148, 2012.

[46] Amanda Vidal. Undecidability and non-axiomatizability of modal many-valued logics. *The Journal of Symbolic Logic*, 87(4):1576–1605, 2022.

[47] Volker Weispfenning. The complexity of the word problem for abelian l-groups. *Theoretical Computer Science*, 48:127–132, 1986.

[48] William Young. Heyting algebras as intervals of commutative, cancellative residuated lattices. Unpublished manuscript, 5 pages, 2014.

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